Singularities and Topological Changes of Deforming Domains in Manifolds

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Abstract. The main purpose of this paper is to investigate when topology jumps while analysis remains continuous. Given a C^0-deformation of domains $D(t)$ in a manifold M^n, allowing the topological type of the domains $D(t)$ to vary with t, in what situations are the analysis notions associated with $D(t)$ continuous in t, so that the machinery of analysis is still working along the deformation? This type of problem arose in our previous work [Hw] for domains on constant mean curvature (CMC) hypersurfaces in $\mathbb{R}^{n+1}$. In the present paper, we consider a general setting in which the deforming domains are situated in an arbitrary smooth manifold M^n equipped with a self-adjoint strongly elliptic operator L, replacing the stability operator for CMC hypersurfaces in $\mathbb{R}^{n+1}$ considered in [Hw].

We introduce the notion of quasi-Lipschitz domains by gluing certain boundary points of a Lipschitz domain in a specific manner, thereby allowing the topology of the deforming domain $D(t)$ to change. The continuity theorems and the existence of the required deformations are proved. We establish that any monotone C^0-deformation of quasi-Lipschitz domains in M^n satisfies Sobolev continuity and eigenvalue continuity for the operator L along the deformation parameter t. As a consequence, a *global* Morse index theorem is obtained. Furthermore, given any quasi-Lipschitz domain D in M^n, we construct a C^0-deformation from a small n-ball to the domain D, along which the topology of $D(t)$ may change, while the required continuity properties remain valid, and the Morse index theorem holds for the deformation.

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§1 Introduction

Let M^n be an oriented *smooth* (i.e., C^∞-) manifold without boundary. A Lipschitz domain D in M^n is a connected open set in M^n, whose closure is compact and whose boundary is Lipschitz.[1].

A *quasi-Lipschitz domain* in M^n is defined by gluing some of the boundary points of a Lipschitz domain in M^n in a prescribed manner (see Definition 2.4 for a precise definition).

Definition 1.1. Given a family

$$\mathcal{D} := \{D(t) \subset M^n \mid 0 \le t \le b\} \tag{1.1}$$

of quasi-Lipschitz domains $D(t)$ in M^n, such that $D(t) \subsetneq D(r)$ for $0 \le t < r \le b$, with

$$D(t) = \bigcup_{s<t} D(s) \quad \text{and} \quad \overline{D(t)} = \bigcap_{r>t} \overline{D(r)}, \tag{1.2}$$

we call $\mathcal{D}$ a *C^0-deformation of enlarging quasi-Lipschitz domains in M^n*, or simply a *monotone C^0-deformation* on M^n. For convenience, we may also simply call $\mathcal{D}$ a *C^0-deformation* on M^n.

The notion of "quasi-Lipschitz domain" is introduced (in Definitions 2.3 and 2.5) for allowing topological changes of $D(t)$ along a given C^0-deformation in t. One of the purposes of this paper is to analyze the relevant Sobolev-theoretic properties near the "singularities" created by the gluing points on its boundary $\partial D(t)$.

For a quasi-Lipschitz domain D in M^n, we consider $L^2(D)$ with the L^2-metric

$$\langle f, g\rangle_{L^2(D)} := \int_D f g \, dM = \int_D f g, \tag{1.3}$$

where dM is the volume form of M^n given by the equipped Riemannian metric on M^n. However, the notation "dM" is usually omitted later, as seen in the last equality. Let

$$\mathcal{F}(D) := \{u \in C^\infty(D) \cap C^1(\overline{D}) \mid u|_{\partial D} = 0\} \tag{1.4}$$

[1]Throughout this paper, we regard M^n as a Riemannian manifold without loss of generality, since (by John Nash) M^n can be equipped with a Riemannian metric by embedding it into an Euclidean space

and let L be a globally defined strongly elliptic second-order operator on M^n such that, for every quasi-Lipschitz domain $D \subset M^n$,

$$L\colon \mathcal{F}(D) \to C^\infty(D) \tag{1.5}$$

is formally self-adjoint in $L^2(D)$, i.e., $\langle Lu, v\rangle_{L^2(D)} = \langle u, Lv\rangle_{L^2(D)}$. Locally, L can be written as

$$Lu = -a^{ij}\partial_i\partial_j u + b^k\partial_k u + cu, \tag{1.6}$$

in which $\partial_i = \frac{\partial u}{\partial x^i}$; $a^{ij} = a^{ij}(x)$, $b^k = b^k(x)$ and $c = c(x)$ are smooth in $x \in D$. Note that $a^{ij} = a^{ji}$, and $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ denotes a local coordinate of D in M^n. By saying that L is strongly elliptic, we mean that

$$a^{ij}\xi_i\xi_j \geq \alpha|\xi|^2, \quad \alpha \text{ a positive constant,} \tag{1.7}$$

for all $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$. It is easy to verify that L of the form (1.6) is formally self-adjoint in $L^2(D)$, if and only if

$$Lu = -\partial_i(a^{ij}\partial_j u) + cu. \tag{1.8}$$

Remark that the strong ellipticity (1.7) is preserved under coordinate changes. The operator L naturally induces a symmetric bilinear form B_L on $\mathcal{F}(D)$ by

$$\begin{aligned} B_L(u, v) &= \int_D (-\partial_i(a^{ij}\partial_j u) + cu)\cdot v \\ &= \int_D Lu\cdot v = \int_D u\cdot Lv. \end{aligned} \tag{1.9}$$

Consider the Sobolev space $W^{1,2}(D)$ consisting of all L^2-functions on D, whose first weak derivatives exist and belong to $L^2(D)$. Equip on the boundary ∂D of D the codimension-one Hausdorff measure induced by the volume form dM of D. Denote

$$\begin{aligned} \mathcal{F}_0(D) &:= \big\{u \in C^\infty(D)\cap C^0(\overline{D});\ u|_{\partial D} = 0\big\}, \\ \mathcal{F}_{cpt}(D) &:= \big\{u \in C^\infty(D);\ \overline{\operatorname{supp} u} \subset D\big\}, \end{aligned} \tag{1.10}$$

where $\overline{\operatorname{supp} u}$ means the closure of the support of u in D. Evidently, $\mathcal{F}_{cpt}(D) \subset \mathcal{F}(D) \subset \mathcal{F}_0(D)$.

Definition 1.2. Given a Lipschitz domain D in M^n, a Sobolev function $g \in W^{1,2}(D)$ satisfies the *boundary L^2-vanishing* condition, if there exist $f_k \in C^\infty(D) \cap C^0(\overline{D})$ such that $\int_{\partial D} f_k^2 \to 0$ and $\|g - f_k\|^2_{W^{1,2}(D)} \to 0$, as $k \to \infty$. We denote the totality of such Sobolev functions by $H(D)$, throughout the present paper.

By Theorem 3.2, which will be proved later in Section 3, the closures of

$$\mathcal{F}_{cpt}(D), \quad \mathcal{F}(D), \text{ and } \quad \mathcal{F}_0(D) \cap W^{1,2}(D)$$

in $W^{1,2}(D)$ all coincide with the same space $H(D)$. Thus $H(D)$ is also the space of Sobolev functions in $W^{1,2}(D)$ with *zero trace* on the boundary ∂D, which is defined as the closure of $\mathcal{F}_0(D) \cap W^{1,2}(D)$ and is denoted by $W_0^{1,2}(D)$ in the literature. Theorem 3.2 asserts that $H(D) = W_0^{1,2}(D) =$ the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$.

Remark 1.1. Let D be the open interval $(0,1)$ in $\mathbb{R}^1$, $u \in \mathcal{F}_0(0,1)$ and $u(x) = \sqrt{x}$ around $x = 0$. Then $u \notin W^{1,2}(0,1)$ since $u'(x) = 1/(2\sqrt{x})$ around $x = 0$ and $u' \notin L^2(0,1)$. Hence, in general, $\mathcal{F}_0(D) \not\subseteq W^{1,2}(D)$.

Extend the symmetric bilinear form B_L given in (1.9) to the Sobolev space $H(D)$, still denoted by $B_L \colon H(D) \times H(D) \to \mathbb{R}^1$, where $\partial_i u$ and $\partial_i v$ in (1.9) are replaced by the weak derivatives $D_i u$ and $D_i v$ respectively, i.e.,

$$B_L(u,v) = \int_D a^{ij} D_j u D_i v + cuv. \tag{1.11}$$

We call $u \in H(D)$ a *Jacobi field* on D if $B_L(u,v) = 0$ for all $v \in H(D)$. For such Sobolev Jacobi fields, the classical equation "$Lu = 0$" in the C^∞-sense is generally meaningless, since second order weak derivatives of u need not exist.

In the Sobolev setting on $H(D)$, we have the spectra of eigenvalues λ_k of B_L:

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_k \le \cdots \to \infty,$$

obtained by the standard variational theory of Rayleigh quotients. Formally, the eigenfunctions u_k satisfy $Lu_k = \lambda_k u_k$, where u_k are the corresponding k-th eigenfunction of L on D. Namely, $B_L(u_k, v) =$

$\lambda_k\langle u_k, v\rangle_{L^2(D)}$, for any $v \in H(D)$, and it is known that

$$\begin{aligned}\lambda_k &= \min\{B_L(u,u) \mid u \in S_k\} \\ &= \min_{V^k}\big\{\max\{B_L(u,u) \mid u \in V^k \cap S\}\big\},\end{aligned} \tag{1.12}$$

where S is the unit sphere $\{u \in H(D) \mid \|u\|_{L^2(D)} = 1\}$ in $H(D)$,

$$S_k := \{u \in S \mid \langle u, u_j\rangle = 0 \text{ for all } j = 1, 2, \ldots, k-1.\}$$

and $V^k \subset H(D)$ denotes k-dimensional linear subspaces of $H(D)$. Furthermore, the eigenfunction u_k attains the minimum of B_L in the first formula of (1.12).

By elliptic regularity theory with Sobolev embedding theorem, the eigenfunctions u_k are regular; in particular, $u_k \in \mathcal{F}_0(D)$. (A similar regularity argument was provided in Remark 2.8 in our previous paper [Hw].)

In particular, a Jacobi field on D defined as above is also in $\mathcal{F}_0(D)$, by the same regularity argument. We may select u_k such that $\{u_1, u_2, \ldots\}$ constitutes an orthonormal basis of $H(D)$. Furthermore, by (1.12), it is straightforward to see that

$$D \subset D' \quad \Longrightarrow \quad \lambda_k(D) \geq \lambda_k(D'), \quad \forall\, k = 1, 2, \ldots, \tag{1.13}$$

where D and D' are two domains in M^n and λ_k are the eigenvalues corresponding to the eigenfunction u_k.

Definition 1.3. We define the *Morse index* $\mathrm{Ind}(D)$ on D to be the maximal dimension of linear subspaces of $H(D)$ on which the bilinear form B_L is negative definite. By the *nullity* $\nu(D)$ of L on D, we mean the dimension of all Jacobi fields on D, i.e., $\nu(D)$ is defined to be the number of linearly independent Jacobi fields on D.

Definition 1.4. We say that the operator L satisfies *the unique continuation property on M^n* if the following holds: Given D, which is a domain (i.e. open and connected) in M^n, and $u \in C^\infty(D)$ with $Lu = 0$. If u vanishes on a nonempty open subset of D, then $u \equiv 0$ on D.

Theorem A (Continuity and Existence)**.** *Let L be a self-adjoint strongly elliptic operator of second order defined on M^n, which satisfies the* *unique continuation property on M^n. For any monotone C^0-deformation $\mathcal{D}$ on M^n (given in Definition 1.1), we have: (i) the Sobolev continuity in t, i.e.,*

$$\overline{\bigcup_{s<t} H_s} = H_t = \bigcap_{r>t} H_r, \tag{1.14}$$

where $H_t := H(D(t))$ for any $t \in [0, b]$, and each H_t is regarded as a subset of H_b, by letting the Sobolev functions in H_t zero on $D(b) \setminus D(t)$; (ii) the eigenvalue continuity, i.e., the eigenvalues $\lambda_k(t) := \lambda_k(D(t))$ of L depend continuously on t. Furthermore, given any quasi-Lipschitz domain D of arbitrary topological type, and a point $p_0 \in D$, there exists a C^0-deformation $\mathcal{D}$ of enlarging quasi-Lipschitz domains $D(t)$ in M^n, $0 \leq t \leq b$, such that $D(0)$ is a small n-ball $B^n(p_0)$ of p_0 in M^n and $D(b)$ coincides with the prescribed domain D.

Indeed, $B^n(p_0)$ stated above is not necessarily an n-ball. It can be a *ball-like neighborhood* of p_0, which means a diffeomorphic image of an n-ball. As a corollary of Theorem A, given two Lipschitz domains D and D' in M^n, there exists a C^0-deformation $\{D(t) \mid a \leq t \leq b\}$ from D' to D with eigenvalues of L continuous in t (by passing through a small n-ball with domains shrinking and then enlarging).

Theorem B. *(A global Morse index theorem) Let $D \subset M^n$ be a Lipschitz domain in M^n and L be given as in Theorem A. The Morse index* $\operatorname{Ind}(D)$ *of L on D can be counted by*

$$\operatorname{Ind}(D) = \sum_{0 \leq t < b} \nu(D(t)), \tag{1.15}$$

where $\mathcal{D} := \{D(t) \subset M^n \mid 0 \leq t \leq b\}$ is a C^0-deformation of enlarging quasi-Lipschitz domains satisfying $D(0) = B^n(p_0)$ of a point $p_0 \in D$ and $D(b) = D$. The topological types of quasi-Lipschitz domains $D(t)$ in $\mathcal{D}$ are allowed to change along the deformation.

Theorems A and B will be proved in the later sections of this paper. The topological types of domains considered in the two theorems may change, for example, from a ball to a ring domain (letting M^2 a cylinder), or to a domain of various topological types. In this sense, we call Theorem B a “global” Morse index theorem, since the deforming domain $D(t)$ can "reach far", i.e. given *any* Lipschitz domain D in M^n, a small ball-like neighborhood $B^2(p_0)$ of any point p_0 in D can be enlarged along the deformation to become D. Note that the eigenvalues $\lambda_k(t)$ of L are still continuous in t, and the Morse index $\mathrm{Ind}(D)$ can be counted by (1.15).

In Smale's earlier contribution [Ss], the Morse index theorem is "local", as the initial domain $D(0)$ and the last one $D(b)$ must be of the same topological type. For example, if M^2 is a cylinder, a ring domain D in M^2, can never be reached starting from a ball-like domain $D(0) \approx B^2(p_0)$ along a smooth deformation. Hereafter, "$\approx$' denotes "diffeomorphic to" throughout this paper.

§2 Quasi-Lipschitz domains

Definition 2.1. A *simple n-domain* is an open set in $\mathbb{R}^n$, diffeomorphic to an n-ball. A *Lipschitz triple* (U, Γ, V) in $\mathbb{R}^n$ is a simple n-domain U in $\mathbb{R}^n$, attached with $\Gamma \subset \partial U$ and a simple (n-1)-domain V in a hyperplane H^{n-1} of $\mathbb{R}^n$, such that (see Figure 2.1)

$$\begin{aligned} &U := \{(x, r) \in \mathbb{R}^n \mid u(x) < r < v(x), x \in V \subset H^{n-1}\}, \\ &\Gamma := \{(x, u(x)) \in \mathbb{R}^n \mid x \in V\}, \\ &V \subset H^{n-1} := \{(x, 0) \in \mathbb{R}^n \mid x = (x^1, x^2, \ldots, x^{n-1})\}, \end{aligned} \tag{2.1}$$

where (x, r) is a chosen coordinate of U in $\mathbb{R}^n$, $u = u(x)$ is a Lipschitz function with Lipschitz constant L_0, i.e.,

$$|u(x) - u(y)| \leq L_0 \cdot |x - y|, \quad \forall\, x, y \in V, \tag{2.2}$$

and $v(x)$ is a smooth function greater than $u(x) \,\forall\, x \in V$. We call V the *parameter set*.

By *a Lipschitz domain D in M^n*, we mean that D is a connected open set in M^n, and for each $p_0 \in \partial D$ there is a coordinate neighborhood (W, ψ) of p_0 in M^n, such that $(\psi(U), \psi(\Gamma), V))$ is a Lipschitz triple in $\mathbb{R}^n$, where $\psi : W \to \mathbb{R}^n$ is a coordinate map of M^n around p_0, $U := D \cap W$, $\Gamma := \partial D \cap W$, and V is a parameter set which is a simple (n-1)-domain in a hyperplane $H^{n-1} \subset \mathbb{R}^n$. We call (U, Γ, V) *a Lipschitz*

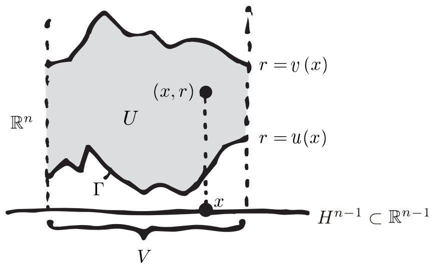


Figure 2.1: Lipschitz triple in $\mathbb{R}^n$

triple of domain D in M^n. For Γ' an open set of ∂D, Γ' is said *Lipschitz* if every $p_0 \in \Gamma'$ has a neighborhood U which is in a Lipschitz triple of domain D in M^n.

Identifying points in W with $\psi(W)$, we may write $p = (x, r)$ for any $p \in U$. When M^n is Riemannian, we can choose the coordinate neighborhood (W, ψ) around $p \in M^n$, so that each r-curve in U of a Lipschitz triple (U, Γ, V) is a geodesic in M^n. This is possible by the standard arguments in elementary differential geometry.

It is clear that every Lipschitz domain D in M^n has a finite family of Lipschitz triples, $(U_1, \Gamma_1, V_1), (U_2, \Gamma_2, V_2), \cdots, (U_h, \Gamma_h, V_h)$, with $\{\Gamma_j; j = 1, 2, \ldots, h\}$ covers ∂D.

Definition 2.2. Let A and B be two sets in $\mathbb{R}^m$ and $\mathbb{R}^\ell$ respectively, not both containing the origin 0 of $\mathbb{R}^{m+\ell} := \mathbb{R}^m \times \mathbb{R}^\ell$, such that the normalization map $\nu\colon A \times B \to S^{m+\ell-1}(1)$, defined by $\nu(x, y) := (x, y)/\sqrt{|x|^2 + |y|^2}$ is injective. Here $S^{m+\ell-1}(1)$ is the unit sphere in $\mathbb{R}^{m+\ell}$. Let $m + \ell = n$. Define the cone with vertex at the origin 0 of $\mathbb{R}^{m+\ell}$ and base $A \times B$ by

$$C_0(A \times B) := \{t(x, y) \in \mathbb{R}^{m+\ell} \mid t \geq 0,\ x \in A,\ y \in B\}. \tag{2.3}$$

For example, we consider $C_0(S^k \times B^{n-k-1})$, where B^n means the unit n-ball, $0 \notin A = S^k$, $0 \in B = B^{n-k-1}$, and $S^k \subset \mathbb{R}^{k+1} = \mathbb{R}^m$, $B^{n-k-1} \subset \mathbb{R}^{n-k-1} = \mathbb{R}^\ell$. Another useful example is $C_0(S^k \times S^{n-k-2})$, where $0 \notin A = S^k \subset \mathbb{R}^m$, and $0 \notin B = S^{n-k-2} \subset \mathbb{R}^\ell$. Both have $m + \ell = n$. In particular, $C_0(S^1 \times B^1)$ and $C_0(S^0 \times S^1)$ in $\mathbb{R}^3$ will be considered later in Example A and Example B respectively.

Definition 2.3. A domain G in M^n is called *diffeomorphic* under a map λ into a cone $C_0(A \times B)$ *relative to* $p \in G$, if (i) $\lambda(p) = 0$, (ii) G is homeomorphic under λ to a neighborhood W of 0 in $C_0(A \times B)$, (iii) $G \setminus \{p\}$ is diffeomorphic to $W \setminus \{0\}$, and given any $(x, y) \in A \times B$, the curve $\lambda^{-1}(t(x, y))$ is differentiable at 0 with nonzero derivative, i.e.,

$$\hat{\nu}(x, y) := \frac{d}{dt}\lambda^{-1}(t(x, y))\Big|_{t=0} \neq 0. \tag{2.4}$$

We are now ready to give a precise definition of *quasi-Lipschitz domains* in M^n. The construction of this definition is somewhat complicated. One may like to consult Example A given after Remark 2.4, while reading the definition.

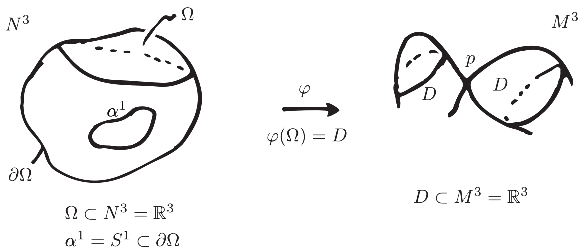


Figure 2.2: gluing map ($n = 3$)

Definition 2.4. Given a domain D in M^n, with its compact closure $\overline{D}$ contained in M^n, i.e. $D \subset\subset M^n$. Consider a map

$$\varphi \colon \Omega \to D, \quad \text{(see Figure 2.2)} \tag{2.5}$$

where $\varphi \in C^\infty(\Omega, M^n) \cap C^0(\overline{\Omega}, M^n)$, Ω is a domain in a smooth manifold N^n with smooth boundary $\partial\Omega$, and $\Omega \subset\subset N^n$. If the following conditions are satisfied, φ is called a *gluing map* of D, and D a *quasi-Lipschitz domain* in M^n:

(i) φ is a diffeomorphism between Ω and D. It can be extended continuously to a neighborhood of Ω in N^n, and $\partial D = \varphi(\partial\Omega)$.

(ii) There exist at most finitely many points $p_1, \ldots, p_\beta \in \partial D$ (called *glued points* of ∂D) such that each $\varphi^{-1}(p_\ell)$ ($\ell = 1, \ldots, \beta$) is either (1) a k-dimensional smooth submanifold α^k of $\partial\Omega$ ($0 < k < n-1$)

which is connected and compact without boundary, or (2) a finite set α^0 of points $q_1, \ldots, q_F$ in $\partial\Omega$. For example, α^k is a k-sphere S^k lying in $\partial\Omega$ for $k \neq 0$ (see Figures 2.2, 2.3), and $\alpha^0 = S^0 = \{q_1, q_2\}$. Note that k depends on p_ℓ. We call α^k the *preglued set* of p_l.

(iii) $\Gamma_0' := \partial\Omega \setminus \varphi^{-1}(\{p_1, \ldots, p_\beta\})$ and $\Gamma_0 := \partial D \setminus \{p_1, \ldots, p_\beta\}$ are homeomorphic under φ. Furthermore, Γ_0 is Lipschitz in the sense that $\forall p \in \Gamma_0$, there is a "Lipschitz triple (U, Γ, V) of the domain D" in M^n (previously defined after Definition 2.1), where $p \in \Gamma = \Gamma_0$ locally.

(iv) Around a glued point $p \in \partial D$,the domain D admits a *sector domain* Λ (see the following (2.6) and Figure 2.5). More precisely, letting S^k and B^m denote the k-sphere and m-ball respectively, there exists a small neighborhood $\widetilde{U}_p$ of p in M^n such that both the two maps

$$\begin{array}{llcl} \lambda\colon & \Lambda := \widetilde{U}_p \cap D & \to & C_0(\alpha^k \times B^{n-1-k}), \\ \lambda|_{\Lambda'}\colon & \Lambda' := \widetilde{U}_p \cap \partial D & \to & C_0(\alpha^k \times S^{n-2-k}), \end{array} \tag{2.6}$$

are diffeomorphic into the cones relative to p with $\lambda(p) = 0$, in the sense of Definition 2.3, and given $x \in \alpha^k$, $y \neq y' \in S^{n-2-k}$,

$$\widehat{\nu}(x, y) \neq \widehat{\nu}(x, y'), \tag{2.7}$$

where $\widehat{\nu}(x, y)$ is the unit vector tangent to the ray $\lambda^{-1}(t(x, y))$ at $t = 0$ with $t \geq 0$, $x \in \alpha^k$ and $y \in S^{n-2-k}$. The domain Λ is called a *sector domain* of D around the glued point p. See Figures 2.3.

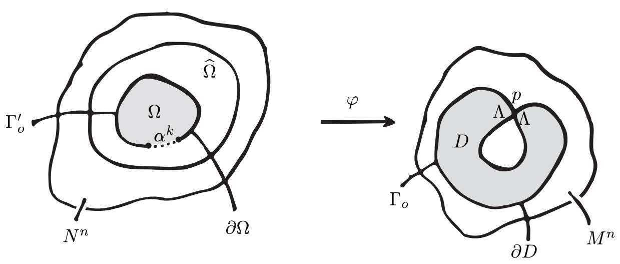


Figure 2.3: a schematic illustration of a quasi-Lipschitz domain

Definition 2.5. For the gluing map $\varphi\colon \Omega \to D$ in Definition 2.4, we say that Ω is the *model domain* of D and N^n is the *model space*. The points $p_1, \ldots, p_\beta \in \partial D$ in Definition 2.4(ii) are called the *glued points* of ∂D.

Definition 2.6. A domain D in M^n is called a *quasi-Lipschitz domain* in M^n if $D = \varphi(\Omega)$ for some gluing map $\varphi\colon \Omega \to D$, where Ω is the model domain in the model space N^n. The glued points of ∂D are also called the *singularities* of ∂D. See Figure 2.3.

Remark 2.1. We distinguish the quasi-Lipschitz domains in M^n for the two cases of the preglued set α^k with $k = 0$ and $k \neq 0$ by the terms "point-glued" and "set-glued".

Remark 2.2. We compare the terminologies in the two papers: In our previous paper [Hw], we defined "generalized Lipschitz domains" where the glued points of ∂D are called "joint points", and they are not necessarily finite and discrete as in this paper. However, no set-glued points are considered in [Hw]. In this paper, we focus especially on set-glued domains, in order to construct C^0-deformation of domains as required in the main theorems, i.e., Theorems A and B.

Definition 2.7. (Properness) For a domain D compactly contained in M^n, we say D is proper, if $\partial D = \partial \overline{D}$.

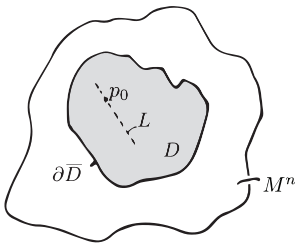


Figure 2.4: $L \not\subset D$, $p_0 \in \partial D \setminus \partial \bar{D}$; D is not proper.

In Figure 2.4, the shaded area D which is an open set of M^n does not contain the line segment L in its interior. Any point $p_0 \in L$ is included in ∂D. But $p_0 \notin \partial \overline{D}$. Thus $p_0 \in \partial D \setminus \partial \overline{D}$, and D is not proper.

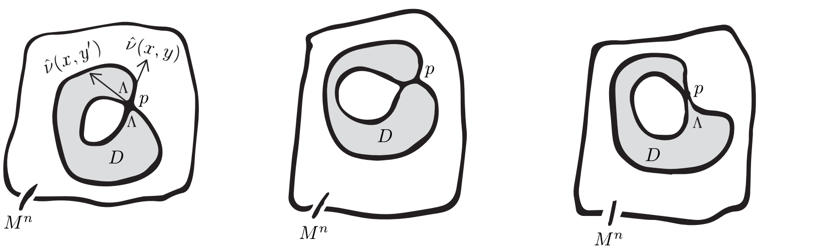
Proposition 2.1. *A quasi-Lipschitz domain is proper.*

Proof. Step 1. It is easy to see that a domain D in M^n is proper, if and only if each boundary point $p \in \partial D$ is a limit point of the exterior $D_c := M^n \setminus \overline{D}$ of D. In fact, if there is $p_o \in \partial D$, such that p_o has a neighborhood N_{p_o} disjoint from D_c , then $N_{p_o} \subset \overline{D}$, which means that $p_o \notin \partial\overline{D}$, and hence $\partial D \neq \partial\overline{D}$. Thus, D is not proper. On the other hand, if D is proper, then every $p_o \in \partial D$ is in $\partial\overline{D}$, since $\partial D = \partial\overline{D}$. It means that p_o is a limit point of D_c, and hence D is proper.

Step 2. A Lipschitz domain D is defined by having a Lipschitz triple around any $p \in \partial D$. Thus, every $p \in \partial D$ is clearly a limit point of D_c, and hence D is proper. A quasi-Lipschitz domain D is basically a Lipschitz domain with a set in ∂D of dimension less than $n-1$ glued together in the manner defined in Definition 2.4. The gluing procedure maintains the property that every $p \in \partial D$ is a limit point of D_c. Therefore, a quasi-Lipschitz domain is still proper. □

Remark 2.3. Indeed, we can define quasi-Lipschitz domains in M^n, by allowing the preimage $\varphi^{-1}(p)$ of a glued point p to consist of a finite union of disjoint k_i-dimensional smooth submanifolds of ∂D with $0 < k_i < n-1$, instead of a single α^k as in Definition 2.4(ii). The main results of this paper carry over to this broader class of domains. However, to avoid the complication of language, we rather restrict $\varphi^{-1}(p)$ to be exactly one α^k when $k \neq 0$.

Remark 2.4. In Figure 2.5(a) and Figure 2.5(b), the shaded area D is a quasi-Lipschitz domain, but it is not the case in Figure 2.5(c), since Λ in the figure is not a sector domain (see Definition 2.4(iv)).



(a) D is quasi-Lip. $\hat{\nu}(x,y) \neq \hat{\nu}(x,y')$

(b) D is quasi-Lip. $\hat{\nu}(x,y) = -\hat{\nu}(x,y')$

(c) D is *not* quasi-Lip. $\hat{\nu}(x,y) = \hat{\nu}(x,y')$: Λ is not a sector domain.

Figure 2.5: examples of quasi-Lipschitz domains

We construct two typical examples to show the essential difference of set-glued and point-glued Lipschitz domains.

Example A. Given $\Omega \approx D \approx$ a 3-ball B^3. Let the preglued set $\alpha^k \approx S^1 \subset \partial\Omega \approx S^2$, where $k = 1$, and the glued point $p \in \partial D \subset \overline{D} \subset M^3 \approx \mathbb{R}^3$, with $\varphi^{-1}(p) = \alpha^1$. Here "$\approx$" means "diffeomorphic to". According to Definition 2.4, $\alpha^1 \subset \partial\Omega$ *cannot* be glued together by "pinching" the circle $\alpha^1 \approx S^1$ into a point p in forming the domain D as in Figure 2.6, since in that case Ω would not be homeomorphic to D under the gluing map φ, violating Definition 2.4(i). In fact, the way to realize the quasi-Lipschitz domain D is not to pinch the neck circle α^1 along the "interior" of Ω. Instead, α^1 should be shrunk and glued to a point p along the "exterior" of Ω in N^3. There are various ways to see this kind of process illustrated in our 3-space: One may regard α^1 as the edge of a dimple (as suggested by the referee of this paper), or regard it as the crater of a volcano.

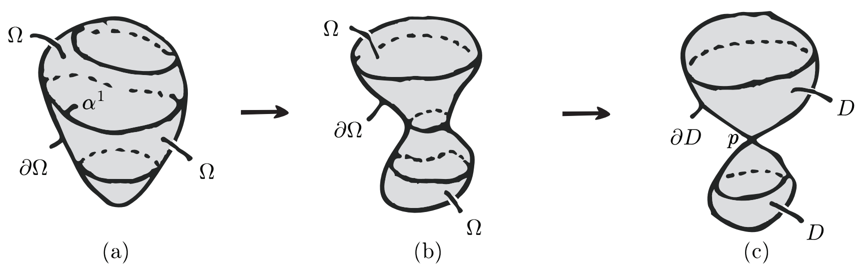


Figure 2.6: Can't pinch preglued set α^1 thru interior to a point

However, we try to visualize it employing "inversion" which exchanges the interior of the 3-ball $B^3 \cong \Omega$ with its exterior, using the standard stereographic projection in the 4-space. In other words, the domain Ω is now the exterior B^3_c of the 3-ball B^3 (subscript "c" means the "complement"), compactified with the point p_∞ "at infinity" (see the shaded area in Figure 2.7(a). Regard the preglued set α^1 as a great circle S^1 of $S^2 \cong \partial\Omega$ ($\approx \partial B^3_c$). The exterior Ω_c of Ω is now the inside ball B^3 bounded by $S^2 \approx \partial\Omega$ (see the white area in Figure 2.7(a)). Pinch $\alpha^1 = S^1$ into the origin 0, which is now the glued point $p \in \partial D \subset \overline{D}$, if the model domain Ω is identified with the glued domain D through

φ, noting that $\Omega \approx D$ by Definition 2.4(i) (see the white area in Figures 2.7(b) and 2.7(c)).

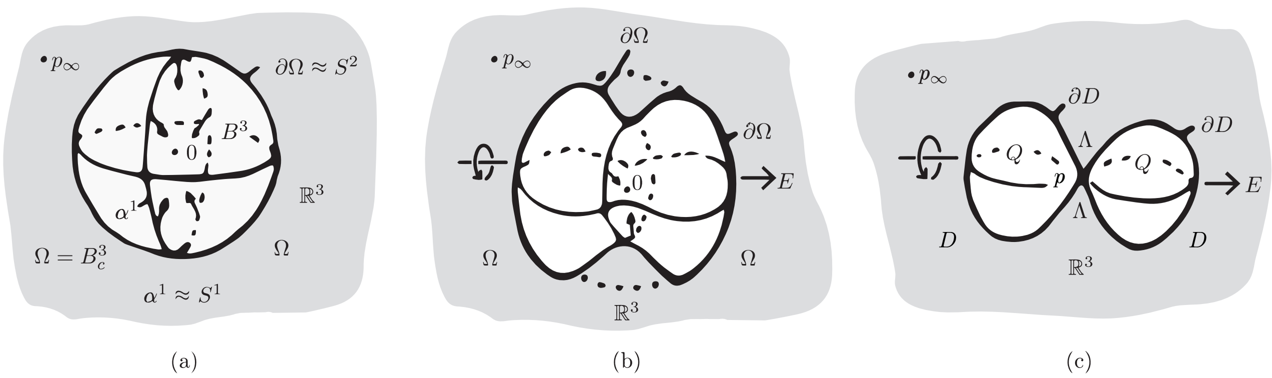


Figure 2.7: Example A: set-glued Lipschitz domain D

The "sector domain" Λ of D around the glued point p has the shape of a domain like $C_0(\alpha^1 \times B^{n-1-1}) = C_0(S^1 \times I)$ (see (2.6)), where $n - 1 - 1 = 0$ and $B^1 = I := (y, y')$, an interval with two end points y and y'. It is obtained basically by rotating along S^1 the triangle $\sigma := C_0(x \times I) \subset \mathbb{R}^2$, $x \in \alpha^1 \approx S^1$ (see Figure 2.8, where (y, y') is denoted by (q_1, q_2)). Let Q denote the domain obtained by rotating the planar area bounded by the digit "8" with the rotation axis E in Figure 2.7(c). Then the required set-glued Lipschitz domain D is $\mathbb{R}^3 \cup \{p_\infty\} \setminus Q$, where the pregued set $\alpha^k = \alpha^1 = S^1$ and the model domain $\Omega \approx$ 3-ball B^3.

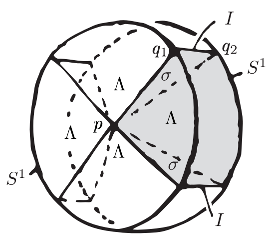


Figure 2.8: sector domain $\Lambda \approx C_0(S^1 \times I)$, obtained by rotating triangle σ along S^1

Example B. The second typical example is a point-glued Lipschitz domain D with $k = 0$, $\alpha^k = \alpha^0 = S^0 = \{q_1', q_2'\} \subset \partial\Omega$. The required point-glued Lipschitz domain D is given by pulling the two points $q_1', q_2' \in \partial\Omega$ in Figure 2.9(a) to the origin 0 of the 3-ball B^3, until the two points are glued together at $p \in \partial D$. Here B^3 means the complement of Ω in $N^3 \approx \mathbb{R}^3$, by the inversion as in Example A (see Figures 2.9(a), 2.9(b) and 2.9(c)). More precisely, if $\widetilde{Q}$ denotes the domain obtained by rotating the planar area bounded by the digit ”8” with rotation axis N in Figure 2.9(c), then D is $\mathbb{R}^3 \cup \{p_\infty\} \setminus \widetilde{Q}$. Here $k = 0, n = 3$, and the sector domain Λ of D around $p \in \partial D$ has the shape of a domain like $C_0(\alpha^0 \times B^{n-1-0}) = C_0(S^0 \times B^2)$ (see (2.6)), which consists of two circular cones joining their vertices at p. (See Figure 2.9(c).)

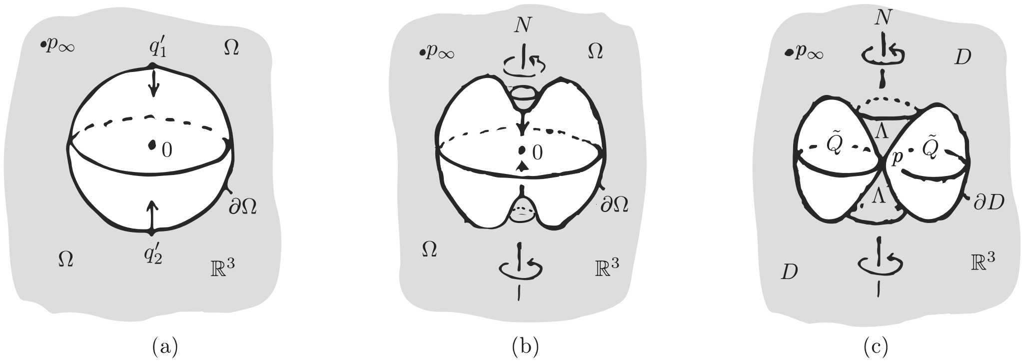


Figure 2.9: Example B: point-glued Lipschitz domain D

Remark 2.5. By Proposition 2.1, any quasi-Lipschitz domain D in M^n is “proper” , i.e., $\partial D = \partial \overline{D}$. In Figure 2.10, the domain D is not proper, since $p \in \partial D \setminus \partial \overline{D}$ is non-empty. The case of Figure 2.10 may happen for a generalized Lipschitz domain (see [Hw]). However, for a quasi-Lipschitz domain $D \subset M^n$, the glued points are assumed finite and discrete in ∂D, and hence D is always proper, as shown in Proposition 2.1.

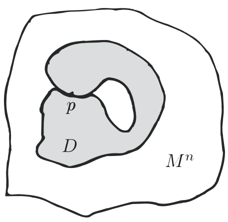


Figure 2.10: non-properness; D is not a quasi-Lipschitz domain

§3 Singularities

Given a Sobolev function $g \in W^{1,2}(M^n)$. By "$g \equiv 0$ on a domain G in M^n", we mean that $g = 0$ almost everywhere on G, or equivalently, $\forall p \in G$, $\exists$ a neighborhood N_p of p, such that $g = 0$ almost everywhere on N_p.

Theorem 3.1 (Trace theorem). *Given a quasi-Lipschitz domain $D \subset\subset M^n$ (i.e., the closure $\overline{D} \subset M^n$) and $g \in W^{1,2}(M^n)$. If $g \equiv 0$ on $D_c := M^n \setminus \overline{D}$, then $g|_D \in H(D)$, and furthermore, $g|_D$ lies in the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$, where*

$$\mathcal{F}_{cpt}(D) := \left\{ h \in C^\infty(D) \mid \overline{\operatorname{supp} h} \subset D \right\}$$

(see also (1.10)). In particular, $H(D) = W_0^{1,2}(D)$, and the trace Tg of g vanishes on ∂D.

Recall that a quasi-Lipschitz domain is proper. For a domain D given as in Figure 2.10 with $g \equiv 0$ on D_c, we note that g may not be in the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$, since D is not proper and neither a quasi-Lipschitz domain.

Corollary 3.1. *Given a quasi-Lipschitz domain $D \subset\subset M^n$, there exists no step function $g \in W^{1,2}(M^n)$ with $g \equiv a$ in D and $g \equiv b$ in D^c, $a \neq b$ being distinct constants.*

In fact, the corollary is a local observation. There is no $g \in W^{1,2}(M^n)$ having a jump on an open subset of ∂D.

To establish the trace theorem, we now enter the jungle of local estimates, and then use the partition of unity to patch together the local results, while treating carefully the behavior of the Sobolev function g around the "singularities" (i.e., the glued points).

Remark 3.1. The trace theorem (Theorem 3.1) is valid for generalized Lipschitz domains defined in [Hw], where the glued points of ∂D (called "joint points" in [Hw]) are not necessarily finite and discrete. Note that in the definition of "generalized Lipschitz domains" in [Hw], we assume the set-continuity which is equivalent to the properness in the present paper. Indeed, Theorem 3.2, Corollaries 3.1 and 3.2 are also valid for generalized Lipschitz domains. The trace theorem developed here will simplify the tedious arguments in [Hw] to obtain the Sobolev continuity there. However, the preglued set α^k with $k \neq 0$ is not allowed for generalized Lipschitz domains in [Hw], and we have to include this difficult case in the trace theorem (Theorem 3.1).

Lemma 3.1 (Local estimates)**.** *Given a Lipschitz triple (U, Γ, V) in M^n, and a Sobolev function $g \in W^{1,2}(U)$. Let $f_k \in C^\infty(U) \cap C^0(\overline{U} \cap \Gamma)$ $(k = 1, 2, 3, \dots)$ with $\int_\Gamma f_k^2 \to 0$ as $k \to \infty$, and*

$$\|f_k - g\|_{W^{1,2}(U)} \to 0 \quad \textit{as } k \to \infty. \tag{3.1}$$

Denote

$$C_\Gamma^\infty(U) := \left\{ h \in C^\infty(U) \mid \overline{\operatorname{supp} h} \cap \Gamma = \varnothing \right\}.$$

Then there exist $h_k \in C_\Gamma^\infty(U)$ such that

$$\|f_k - h_k\|^2_{W^{1,2}(U)} \to 0 \quad \textit{as } k \to \infty, \tag{3.2}$$

and therefore

$$\|h_k - g\|^2_{W^{1,2}(U)} \to 0 \quad \textit{as } k \to \infty.$$

Proof. Step 1. Use the notations for (U, Γ, V) in (2.1) and (2.2). Let $\overline{\delta}$ be a positive constant, define the $\overline{\delta}$-tubular neighborhood

$$N_{\overline{\delta}} := \{(x, r) \in U \mid x \in V, u(x) < r < u(x) + \overline{\delta}\}.$$

Put $\overline{\delta} = 4\delta$. Consider $\overline{u} := u + 2\delta$ and the smooth approximation function $w(x)$ of $\overline{u}(x)$ given by (see Figure 3.1)

$$w(x) := \int_V \overline{u}(y) \cdot \varphi_\alpha(y - x)\, dy, \quad \forall\, x \in V,$$

where $\alpha > 0$ is a small number, and

$$\varphi_\alpha(x) := \frac{1}{\alpha^{n-1}} \varphi\left(\frac{x}{\alpha}\right)$$

with $\varphi \in C^\infty(H^{n-1})$ the standard mollifier. (We may freely expand U and V in order to make $w(x)$ well-defined around ∂V in $H^{n-1} \subset \mathbb{R}^{n-1}$.) For $x \in V$, we see that

$$|w(x) - \overline{u}(x)| = \left| \int_V (\overline{u}(y) - \overline{u}(x)) \varphi_\alpha(y - x)\, dy \right| \leq L_0 \cdot \alpha = \delta,$$

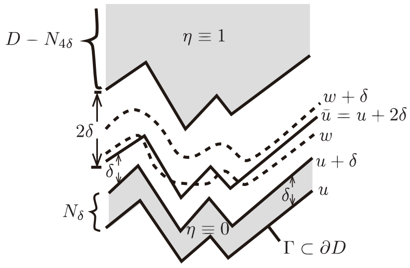


Figure 3.1: local estimates

by choosing $\alpha = \delta/L_0$, where L_0 is the Lipschitz constant for $u = u(x)$.

Step 2. Define $\eta \in C^\infty(U)$ by $\eta(x, r) := \eta_0\left(\frac{r-w(x)}{\delta}\right)$, where $\eta_0 \in C^\infty(\mathbb{R})$ is a cut-off function with

$$\begin{cases} \eta_0(s) = 0 & \text{for } s \leq 0, \\ 0 \leq \eta_0(s) \leq 1 & \text{for } 0 \leq s \leq 1, \\ \eta_0(s) = 1 & \text{for } s > 1. \end{cases}$$

Clearly, $\eta = 0$ on N_δ and $\eta = 1$ on $U \setminus N_{\overline{\delta}}$, recalling $\overline{\delta} = 4\delta$. Define

$$h_k := f_k \cdot \eta \in C^\infty_\Gamma(U). \tag{3.3}$$

We start to estimate the Sobolev norm of $f_k - h_k$:

$$\int_U |f_k - h_k|^2 = \int_{N_{\overline{\delta}}} |f_k(1-\eta)|^2 \leq \int_{N_{\overline{\delta}}} f_k^2 \leq C\left(\int_{N_{\overline{\delta}}} |Df_k|^2\right) \cdot \delta^2, \tag{3.4}$$

in which the last inequality will be shown later by (3.7) and Step 5;

$$\begin{aligned} \int_U |Df_k - Dh_k|^2 &= \int_{N_{\overline{\delta}}} |Df_k(1-\eta) - f_k(D\eta)|^2 \\ &\leq C\left(\int_{N_{\overline{\delta}}} |Df_k|^2 + \int_{N_{\overline{\delta}}} f_k^2 |D\eta|^2\right), \end{aligned} \tag{3.5}$$

where C in the formulas denotes positive constants for *any* form, independent of δ, x, r and $f_k, h_k, \ldots$, such as $C = 2C = 2 = L_0 = 1 + L_0^2 = \cdots$, etc. Sometimes, we use $C_1, C_2, \ldots$, etc., when it is necessary to distinguish them. Let the last term of (3.5) be denoted by

$$W := C \int_{N_{\overline{\delta}}} f_k^2 \cdot |D\eta|^2, \tag{3.6}$$

which will be estimated later.

Step 3. $|D\eta|^2 \leq C_1/\delta^2$ is to be established later in (3.13). On the other hand, we shall show later in Step 5 that

$$\int_{N_{\bar{\delta}}} f_k^2 \leq C(\int_\Gamma f_k^2)\cdot\delta + C(\int_\Gamma f_k^2)^{1/2}\cdot P_k(\delta)^{1/2}\cdot\delta^{3/2} + C\cdot P_k(\delta)\cdot\frac{\delta^2}{2}$$
$$\leq C\cdot Q_k(\delta)\cdot\delta + C\cdot P_k(\delta)\cdot\delta^2, \tag{3.7}$$

where

$$Q_k(\delta) := \int_\Gamma f_k^2 + (\int_\Gamma f_k^2)^{1/2}\cdot P_k(\delta)^{1/2}\cdot\delta^{1/2}, \tag{3.7a}$$

$$P_k(\delta) := \int_{N_{\bar{\delta}}} |Df_k|^2 \leq \int_{N_{\bar{\delta}}} |Dg|^2 + \alpha_k \tag{3.7b}$$

and $\alpha_k \to 0$ as $k \to \infty$ by (3.1). Then we shall have

$$W \leq C(\int_{N_{\bar{\delta}}} f_k^2)\cdot\frac{C_1}{\delta^2} \leq C\cdot Q_k(\delta)\cdot\frac{1}{\delta} + CP_k(\delta). \tag{3.8}$$

Combining (3.4), (3.5), (3.6) and (3.8), we have

$$\|f_k - h_k\|^2_{W^{1,2}(U)} \leq C_1\cdot Q_k(\delta)\cdot\frac{1}{\delta} + C_2P_k(\delta), \tag{3.9}$$

by choosing a small $\delta = \delta_0 < 1$. This will prove (3.2), as required. More precisely, for any $\epsilon > 0$, choose $\delta = \delta_0 < 1$ such that

$$\int_{N_{\bar{\delta}}} |Dg|^2 < \frac{\epsilon}{4C_2}.$$

Then selecting k_0 such that for $k > k_0$, $\alpha_k < \frac{\epsilon}{4C_2}$ is valid, where α_k is given in (3.7b), we obtain that $C_2P_k(\delta_0) < \frac{\epsilon}{2}$. By $\int_\Gamma f_k^2 \to 0$ as $k \to \infty$, there exists $k_1 > k_0$ such that for $k > k_1$,

$$\frac{1}{\delta_0}Q_k(\delta_0) < \frac{\epsilon}{2C_1}. \tag{3.10}$$

Hence for $\delta = \delta_0$, $k > k_1$, we obtain from (3.9) that

$$\|f_k - h_k\|^2_{W^{1,2}(U)} < \epsilon.$$

A *key* point of the above estimation is that we *squeeze* out δ^2 of $\int_{N_{\bar\delta}} f_k^2$ to compensate the factor $1/\delta^2$ arising from the steepness of $|D\eta|^2$ as δ gets very small.

Step 4. We have

$$\begin{aligned}|D\eta|^2 &\le 2(|D_x\eta|^2 + (\frac{\partial\eta}{\partial r})^2)\\ &= 2[\eta_0'(\frac{r-w(x)}{\delta})]^2 \cdot [(\frac{-1}{\delta}|D_x w|)^2 + (\frac{1}{\delta})^2]\\ &\le 2\,\frac{C_3}{\delta^2}\,(1+|D_x w|^2),\end{aligned} \tag{3.11}$$

where C_3 is the bound of the term $|\eta_0'|^2$, a constant independent of δ. In the first inequality of (3.11), the number 2 is attributed to the deviation of the Riemannian metric g_{ij} of M^n from the flat metric (x,r) of (U,Γ,V). Claim that

$$|D_x w| \le L_0. \tag{3.12}$$

Given $x \in V$, for z very close to x in V, we have

$$\begin{aligned}|w(z)-w(x)| &= \left|\int_V \overline{u}(y)\varphi_\alpha(y-z)\,dy - \int_V \overline{u}(y)\varphi_\alpha(y-x)\,dy\right|\\ &= \left|\int_{B_\alpha(0)} \overline{u}(z+\zeta')\varphi_\alpha(\zeta')\,d\zeta' - \int_{B_\alpha(0)} \overline{u}(x+\zeta)\varphi_\alpha(\zeta)\,d\zeta\right|\\ &\le \int_{B_\alpha(0)} \left|\overline{u}(z+\zeta) - \overline{u}(x+\zeta)\right|\,\varphi_\alpha(\zeta)\,d\zeta \le L_0\cdot|z-x|.\end{aligned}$$

Letting $z \to x$, we obtain (3.12) and hence

$$|D\eta|^2 \le 2C_3\,\frac{1+L_0^2}{\delta^2} = \frac{C}{\delta^2} \tag{3.13}$$

by (3.11).

Step 5. Fix k, denote $f := f_k$ given in (3.1), in order to simplify the notations in the following computations. Claim (3.7): Note that $f = f(x,r)$, $x \in V$ and $u(x) < r < u(x) + \overline{\delta}$. Let $\overline{r} = r - u(x)$. But we still write r, replacing $\overline{r}$ for convenience. It is clear to see that

$$
\begin{aligned}
&\int_{N_{\overline{\delta}}} f^2\, dr dx = \int_V (\int_0^{\overline{\delta}} f(x,r)^2\, dr)\, dx \\
&= \int_V (\int_0^{\overline{\delta}} [f(x,0) + \int_0^r f_s(x,s)\, ds]^2\, dr)^2\, dx \\
&= \int_V (\int_0^{\overline{\delta}} [f(x,0)^2 + 2f(x,0) \int_0^r f_s(x,s)\, ds + (\int_0^r f_s(x,s)\, ds)^2]\, dr\, dx,
\end{aligned}
\tag{3.14}
$$

where $f_s := \frac{\partial f}{\partial s}(x,s)$. We simplify the computations by letting the area form $= dr \wedge dx$, disregarding the difference $\omega := dM - dr \wedge dx$. Evidently, the simplification make sense, because we only focus on the order of δ, to which the deviation ω is negligible.

Denote the three terms in (3.14) by α, β and γ, respectively. Then

$$
|\alpha| = \int_V (\int_0^{\overline{\delta}} (f(x,0))^2\, dr)\, dx = C(\int_\Gamma f^2) \cdot \overline{\delta},
$$

$$
\begin{aligned}
|\beta| &\le 2 \int_V |f(x,0)| (\int_0^{\overline{\delta}} \left| \int_0^r f_s(x,s)\, ds \right| dr)\, dx \\
&\le 2 \int_V |f(x,0)| (\int_0^{\overline{\delta}} (\int_0^r f_s(x,s)^2\, ds)^{1/2} (\int_0^r 1^2\, ds)^{1/2}\, dr)\, dx \\
&\le 2 \int_V |f(x,0)|\; (\int_0^{\overline{\delta}} |Df|^2\, ds)^{1/2} \cdot \frac{2}{3}\, \overline{\delta}^{3/2}\, dx \\
&\le \frac{4}{3}\, \overline{\delta}^{3/2} (\int_V f(x,0)^2\, dx \cdot \int_V \int_0^{\overline{\delta}} |Df|^2\, ds\; dx)^{1/2} \\
&= C\; \delta^{3/2} (\int_\Gamma f^2)^{1/2} (\int_{N_{\overline{\delta}}} |Df|^2)^{1/2} = C\; \delta^{3/2} (\int_\Gamma f^2)^{1/2} \cdot P_k(\delta)^{1/2},
\end{aligned}
$$

$$
\begin{aligned}
|\gamma| &= \int_V \int_0^{\overline{\delta}} (\int_0^r f_s(x,s)\, ds)^2\, dr dx \le \int_V \int_0^{\overline{\delta}} (\int_0^r |Df|^2\, ds)\, r\, dr dx \\
&\le \int_V (\int_0^{\overline{\delta}} |Df|^2\, ds \cdot \int_0^{\overline{\delta}} r\, dr)\, dx = C\, \frac{\overline{\delta}^2}{2}\, P_k(\delta).
\end{aligned}
$$

Thus we obtain (3.7) and complete the proof of Lemma 3.1. □

We consider the simplest case of $n = 1$ to review the essence of the above proof. The key point is in Step 3 of the proof that we estimated

$\int_{N_\delta} f_k^2 = o(\delta^2)$ to compensates the order c/δ^2 of $|D\eta|^2$. Write $f := f_k$. Let $f(x) = x$, with D:=(0,1), then $\int_{N_\delta} f^2 = \int_0^\delta x^2 dx = [x^3/3]_0^\delta = \delta^3/3$. Here, $f \in W^{1,2}(D)$, and the crucial term W in (3.6) satisfies

$$W := C \int_{N_\delta} f^2 \cdot |D\eta|^2 = C \cdot \delta^3/3 \cdot C_1/\delta^2 = C_2\,\delta, \tag{3.15}$$

which tends to zero, as δ tends to zero.

For a negative example, let $f(x) = \sqrt{x}$, $x \in (0,1)$, then f is not a Sobolev function, since $f'(x) = 1/\sqrt{x}$, and $\int_0^\delta [f'(x)]^2\,dx = \infty$. (Recalling Remark 1.1). Hence $f(x)$ does not meet (3.1), which requires in this case that $\int_0^\delta |f'(x) - Dg(x)|^2\,dx$ is arbitrarily small.

In order to figure out the intuitive meaning of the estimation that $\int_{N_\delta} f^2 = o(\delta^2)$, one may expand $f(x)$ into

$$f(x) = f(0) + f'(0)x + E(x), \quad \text{with } f(0) = 0,$$

and $E(x) = o(x)$, to find that

$$\begin{aligned}\int_0^\delta [f(x)]^2\,dx &= \int_0^\delta (f'(0)x + E(x))^2\,dx \\ &= [f'(0)]^2 \int_0^\delta x^2\,dx + 2f'(0)\int_0^\delta xE(x)\,dx + \int_0^\delta [E(x)]^2\,dx \\ &= O(\delta^3),\end{aligned}$$

which is analogous to (3.14) and (3.7).

Theorem 3.2 (Global version)**.** *Given a quasi-Lipschitz domain D in M^n, let g be a Sobolev function in $H(D)$, i.e. g satisfies the boundary L^2-vanishing condition defined by Definition 1.2. Then g is in the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$ (see (1.10)). In particular, the trace Tg of g on ∂D vanishes, i.e. $H(D) = W_0^{1,2}(D)$.*

Recall some concepts: The hypothesis of $g \in H(D)$ means the existence of $f_k \in C^\infty(D) \cap C^0(\overline{D})$ $(k = 1,2,3,\ldots)$ with $\int_{\partial D} f_k^2 \to 0$ and $\|f_k - g\|_{W^{1,2}(D)} \to 0$ as $k \to \infty$. Given g in $W^{1,2}(D)$, the *trace* of g on ∂D is defined by the boundary value $u \in C^0(\partial D)$, such that there exist $f_k \in C^\infty(D) \cap C^0(\overline{D})$ with $f_k|_{\partial D} = u$, and $||f_k - g||^2_{W^{1,2}(D)} \to 0$, as $k \to \infty$. Particularly, $W_0^{1,2}(D)$ consists of functions in $W^{1,2}(D)$ with zero trace on ∂D. Namely, $W_0^{1,2}(D)$ is the closure of $\mathcal{F}_0(D) \cap W^{1,2}(D)$ in $W^{1,2}(D)$.

As a corollary of Theorem 3.2, we have the following closure theorem.

Corollary 3.2 (Closure theorem)**.** *Given a quasi-Lipschitz domain D in M^n, the Sobolev space $H(D)$ defined by Definition 1.2 is identical with each of the closures of*

$$\mathcal{F}_{cpt}(D), \quad \mathcal{F}(D), \ \text{and} \quad \mathcal{F}_0(D) \cap W^{1,2}(D)$$

in $W^{1,2}(D)$.

We shall complete the proofs of Theorem 3.1 and 3.2 , by treating the behavior of a Sobolev function $g \in W^{1,2}(D)$ around the glued points of ∂D, which are the singularities of the quasi-Lipschitz domains. The two theorems pave a way to establish the Sobolev continuity and eigenvalue continuity.

Proof of the trace theorem (Theorem 3.1). Assume that the next Theorem 3.2 is valid. For $g \in W^{1,2}(M^n)$ with g and D given in Theorem 3.1, there exist $f_k \in C^\infty(M^n)$, $k = 1, 2, 3, \ldots$, such that $\|f_k - g\|_{W^{1,2}(M^n)} \to 0$ as $k \to \infty$. We claim that $\int_{\partial D} f_k^2 \to 0$ as $k \to \infty$.

Step 1. Given $p \in \partial D$ (p may be a glued point of ∂D). We consider an n-ball B_p in M^n, centered at p and the set $G := B_p \cap D_c$, where $\partial G = \Gamma_p \cup K$, $\Gamma_p := \partial D \cap \partial G$, $K := \partial G \cap D_c$ (see Figure 3.2).

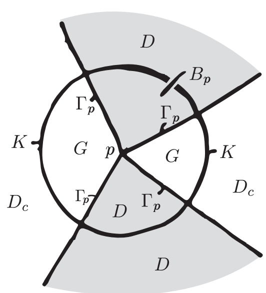


Figure 3.2: to estimate $\int_{\partial D} f_k^2$

We use the method of moving frames to compute the relevant integrals. Let $\omega^1, \omega^2, \ldots, \omega^n$ be an orthonormal coframe in G, and dA the area form of ∂G (with slight smooth modification). The volume form dM of G is $dM = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n$. Remark that

$$\begin{aligned} dA &= a_1\, \widehat{\omega^1} \wedge \omega^2 \wedge \cdots \wedge \omega^n + a_2\, \omega^1 \wedge \widehat{\omega^2} \wedge \cdots \wedge \omega^n + \cdots \\ &\qquad + a_n\, \omega_1 \wedge \omega^2 \wedge \cdots \wedge \omega^{n-1} \wedge \widehat{\omega^n}, \\ df_k &= \partial_i f_k\, \omega^i \quad \text{(using summation convention)}, \end{aligned}$$

where " $\widehat{**}$ " means "absence".

Step 2. We have

$$\begin{aligned} \int_{\Gamma_p} f_k^2\, dA &\le \int_{\Gamma_p \cup K} f_k^2\, dA = \int_{\partial G} f_k^2\, dA = \int_G d(f_k^2\, dA) \quad \text{(by Stoke's theorem)} \\ &= \int_G df_k^2 \wedge dA + \int_G f_k^2\, ddA = \int_G \left[\partial_1(f_k^2)\omega^1 + \cdots\right] \wedge dA + \int_G f_k^2\, ddA \\ &= \int_G \left[2f_k(\partial_1 f_k)a_1 + \cdots\right]\omega^1 \wedge \cdots \wedge \omega^n + \int_G f_k^2 (\sum_i a_{ii})\omega^1 \wedge \cdots \wedge \omega^n \\ &\le C_1 \left(\int_G f_k^2\right)^{1/2} \left(\int_G |Df_k|^2\right)^{1/2} dM + C_2 \int_G f_k^2\, dM \\ &\le\ C\|f_k\|^2_{W^{1,2}(G)} \le\ C\|f_k - g\|^2_{W^{1,2}(G)} \quad (\text{since } g \equiv 0 \text{ in } G \subset D_c) \\ &\to 0 \quad \text{as } k \to \infty. \end{aligned}$$

Applying Theorem 3.2 to the given $g \in W^{1,2}(M^n)$, we see that the trace Tg of g on ∂D vanishes, and the trace theorem is established. □

Proof of Theorem 3.2. *Step* 1. Let $g \in H(D)$ and $f_k \in C^\infty(D) \cap C^0(\overline{D})$ be given in Definiton 1.2. Around each glued point $p \in \partial D$, either point-glued or set-glued, we consider a neighborhood U_p of p in D, and let

$$\begin{aligned} r = r(x) &:= \mathrm{dist}_{M^n}(x, p), \text{ with } r(p) = 0; \\ U_p \supset\supset U_{4\delta} &:= \{x \in D : 0 < r(x) < 4\delta\}, \end{aligned} \tag{3.16}$$

where $\delta > 0$ is a small number to be determined later. Remark that the neighborhood U_p could be of the shape like the Figure 2.8 in Example A. Locally around p, we identify U_p with an open neighborhood in $C^0(\alpha^\ell \times$

$B^{n-1-\ell}$) through the map λ in (2.6) of Definition 2.4. Parametrize $x \in U_p$ by $x = (r, \xi)$, where $r > 0$ and $\xi \in$ A:= an open set in $\alpha^\ell \times S^{n-1-\ell}$. Define on U_p the cut-off function $\eta \in C^\infty(U_p) \cap C^0(\overline{U}_p)$ by

$$\eta(x) := \begin{cases} 0, & \text{for } 0 \leq r(x) \leq \delta, \\ 1, & \text{for } 4\delta \leq r(x), \end{cases} \tag{3.17}$$

where η is increasing in r and $|D\eta| < \frac{1}{6\delta}$ on U_p. Given $k > 0$, we define

$$\widehat{h}_k := f_k \eta \in C^\infty(U_p) \cap C^0(\overline{U}_p). \tag{3.18}$$

Evidently, $\widehat{h}_k \equiv 0$ on $U_\delta := \{x \in D : 0 < r(x) < \delta\}$, and $\overline{\text{supp } \widehat{h}_k} \cap \Gamma_\delta = \varnothing$, where $\Gamma_\delta := \overline{U}_\delta \cap \partial D = \{x \in \partial D : 0 \leq r(x) \leq \delta\}$.

Step 2. Given $\varepsilon > 0$, we claim that $\|f_k - \widehat{h}_k\|^2_{W^{1,2}(U_{4\delta})} < \frac{\varepsilon}{3\beta}$, β being the number of glued points of ∂D for $\delta =$ some $\delta_0 > 0$ and $k >$ some $k_0 > 0$ (see (iii) in Definition 2.4). The proof is similar to the proof of Lemma 3.1, except now the r-curves are radiating from the glued point $p \in \partial D$, at which $r(p) = 0$.

(i) To estimate $\|f_k - \widehat{h}_k\|^2_{W^{1,2}(U_{4\delta})} < \frac{\varepsilon}{3\beta}$, we see that

$$\int_{U_{4\delta}} |f_k - \widehat{h}_k|^2 \leq \int_{U_{4\delta}} f_k^2 = \int_{U_{4\delta}} g^2 + \alpha_k, \tag{3.19}$$

and

$$\begin{aligned} \int_{U_{4\delta}} |Df_k - D\widehat{h}_k|^2 &\leq C\left(\int_{U_{4\delta}} |Df_k|^2 + W\right) \\ &= C\left(\int_{U_{4\delta}} |Dg|^2 + \beta_k\right) + CW, \end{aligned} \tag{3.20}$$

where $\alpha_k \to 0$, $\beta_k \to 0$ as $k \to \infty$, and

$$W := C \int_{U_{4\delta}} f_k^2 \, |D\eta|^2 < \frac{\varepsilon}{9\beta} \quad \text{on } U_{4\delta}. \tag{3.21}$$

Remark that (3.19) and (3.20) are based on the computations in (3.4) and (3.5).

(ii) The equality in (3.19) is elementary. It derives simply from Cauchy-Schwarz inequality. However, we will compute it explicitly as follows:

$$\int_{U_{4\delta}} f_k^2 = \int_{U_{4\delta}} g^2 + (f_k^2 - g^2) = \int_{U_{4\delta}} g^2 + \alpha_k,$$

where

$$
\begin{aligned}
|\alpha_k| &= \left| \int_{U_{4\delta}} (f_k - g)(f_k + g) \right| \\
&= \left| \int_{U_{4\delta}} (f_k - g)\big(2g + (f_k - g)\big) \right| \\
&\le 2 \left(\int_{U_{4\delta}} |f_k - g|^2 \int_{U_{4\delta}} g^2 \right)^{1/2} + \int_{U_{4\delta}} |f_k - g|^2 \\
&= \left(\int_{U_{4\delta}} |f_k - g|^2 \right)^{1/2} \left[2 \left(\int_{U_{4\delta}} g^2 \right)^{1/2} + \left(\int_{U_{4\delta}} |f_k - g|^2 \right)^{1/2} \right] \longrightarrow 0
\end{aligned}
\tag{3.22}
$$

as $k \to \infty$. Similarly, $\int_{U_{4\delta}} |Df_k|^2 = \int_{U_{4\delta}} |Dg|^2 + \beta_k$, with $\beta_k \to 0$ as $k \to \infty$.

(iii) It is not difficult to see that $|D\eta|^2 \le C_1/\delta^2$, which is similar to the proof of (3.13). Furthermore,

$$
\begin{aligned}
W := C \int_{U_{4\delta}} f_k^2 \, |D\eta|^2 &\le C \left(\int_{U_{4\delta}} f_k^2 \right) \frac{C_1}{\delta^2} \\
&= C \left[\left(\int_{U_{4\delta}} g^2 \right) + \alpha_k \right] \frac{C_1}{\delta^2} \\
&= C\big(C_2 \cdot \mathrm{vol}(U_{4\delta}) + \alpha_k\big) \frac{C_1}{\delta^2} \\
&= C_3 \delta^{n-2} + C C_1 \alpha_k \cdot \frac{1}{\delta^2}.
\end{aligned}
\tag{3.23}
$$

Suppose $n := \dim M^n > 2$. We will see (3.21) immediately: For the given $\varepsilon > 0$, we can find $\delta_0 > 0$ such that $C_3 \delta^{n-2} < \frac{\varepsilon}{18\beta}$, for $0 < \delta \le \delta_0$. Then choose $k_0 > 0$ such that

$$
C C_1 \alpha_k \cdot \frac{1}{\delta_0^2} < \frac{\varepsilon}{18\beta}, \quad \text{for } k > k_0. \tag{3.24}
$$

Thus there exist $\delta_0 > 0$ and $k_0 > 0$ such that

$$
W \le \frac{\varepsilon}{9\beta} \quad \text{on } U_{4\delta_0}, \quad \text{for } k > k_0. \tag{3.25}
$$

(iv) Remark that both $\int_{U_{4\delta}} g^2$ and $\int_{U_{4\delta}} |Dg|^2$ are of the order $O(\delta^n)$. The argument in (iii) can be applied to the two corresponding terms in (3.19) and (3.20) to find the common $\delta_0 > 0$ and $k_0 > 0$, such that these terms in the right hand sides of (3.19) and (3.20) are less than $\frac{\varepsilon}{9\beta}$. Now define globally a function $\widetilde{f}_k \in C^\infty(D) \cap C^0(\overline{D})$ such that $\widetilde{f}_k := \widehat{h}_k$ on each U_p, where $p \in \partial D$ is a glued point, and $\widetilde{f}_k := f_k$ elsewhere in D. Then for $n := \dim M^n > 2$, we have

$$\|f_k - \widetilde{f}_k\|^2_{W^{1,2}(D)} < \frac{\varepsilon}{3}, \tag{3.26}$$

and $\overline{\operatorname{supp}\ \widetilde{f}_k} \cap \Gamma_{\delta_0} = \varnothing$, around each glued point $p \in \partial D$.

(v) As for $n = 2$, the proof of (3.26) is reduced to Lemma 3.1. In fact, consider a neighborhood U_p of a glued point $p \in \partial D$. The boundary segment $\Gamma := \overline{U}_p \cap \partial D = \Gamma_1 \cup \cdots \cup \Gamma_F$ (see (2) of (ii) in Definition 2.4), where $\Gamma_i \cap \Gamma_j = \{p\}$,$\forall\, i \neq j$. Furthermore, each Γ_i is a simple Lipschitz line, which is provided by requirement (v) of Definition 2.4 concerning the sector domains. Hence there are Lipschitz triples (U_i, Γ_i, V_i) for which Lemma 3.1 can be applied to obtain $\widehat{h}_k$ approximating f_k in $W^{1,2}(U_i)$. Define a global $\widetilde{f}_k \in C^\infty(D) \cap C^0(\overline{D})$ as in (iv) of this step, i.e., Step 2. We obtain (3.26).

Step 3. Consider $\Gamma' := \bigcup\{\Gamma_{\delta_0} : p \text{ is a glued point on } \partial D\}$, where $\delta_0 > 0$ is the common small number chosen for all glued points $p \in \partial D$, and $\Gamma_{\delta_0} := \overline{U}_{\delta_0} \cap \partial D$. Then Γ' is a compact Lipschitz set contained in ∂D. There exists a finite cover Γ'_j, $j = 1, 2, \ldots, \mu$, with the corresponding Lipschitz triples (U'_j, Γ'_j, V'_j). Remark that $\int_{\partial D} \widetilde{f}_k^2 \longrightarrow 0$ as $k \to \infty$. We can apply Lemma 3.1 to these triples to obtain $h'_j \in C^\infty(U'_j) \cap C^0(\overline{U}'_j)$ with $\overline{\operatorname{supp}\ h'_j} \cap \Gamma'_j = \varnothing$, such that h'_j is sufficiently close to $\widetilde{f}_k$ in $W^{1,2}(U'_j)$. Use the standard partition of unity (see Section 3.4 in [Hw] for details) to patch together the local approximations $\{h'_j \text{ on } U'_j\,;\, j = 1, 2, \ldots, \mu\}$, and $\widetilde{f}_k$ on $G_0 := D \setminus \bigcup\{U'_j : j = 1, 2, \ldots, \mu\}$ to obtain $h_k \in C^\infty(D) \cap C^0(\overline{D})$, such that $\operatorname{supp} h_k \subset\subset D$ and

$$\|\widetilde{f}_k - h_k\|^2_{W^{1,2}(D)} < \frac{\varepsilon}{3}. \tag{3.27}$$

Finally, we choose $k_0 > 0$, such that for $k > k_0$,

$$\|f_k - g\|^2_{W^{1,2}(D)} < \frac{\varepsilon}{3}, \tag{3.28}$$

by the assumption of Theorem 3.2. Combining (3.26), (3.27), and (3.28), we have

$$\|g - h_k\|^2_{W^{1,2}(D)} < \varepsilon,$$

where $h_k \in \mathcal{F}_{\mathrm{cpt}}(D)$ and k sufficiently large. In other words, $g \in W_0^{1,2}(D)$. This completes the proof. □

Remark 3.2. In the above Step 3 (of the proof of Theorem 3.2, we use the partition of unity to patch together the functions $\{h'_j$ on U'_j; $j = 1, 2, \dots, \mu\}$ and $\{\tilde{f}_k$ on $G_0\}$, where $G_0 := D \setminus (U'_1 \cup U'_2 \cup \cdots \cup U'_\mu)$. Let us explain this construction in more detail as follows. Let the shrunk open set U'_j be denoted by G_j so that $\{G_0, G_1, \dots, G_\mu\}$ covers D. Let $\{\varphi_0, \varphi_1, \dots, \varphi_\mu\}$ be the partition of unity on G_j, $j = 0, 1, \dots, \mu$, i.e.,

$$1 = \varphi_0 + \varphi_1 + \cdots + \varphi_\mu,$$

where $\overline{supp\,\varphi_j} \subset G_j$. Write $\tilde{f} := \tilde{f}_k$ on G_0. Consider the integrated function

$$h_k := \varphi_0 \tilde{f} + (\varphi_1 h'_1 + \varphi_2 h'_2 + \cdots + \varphi_\mu h'_\mu).$$

Then

$$\begin{aligned}
\|\tilde{f} - h_k\|^2_{W^{1,2}(D)} &= \sum_{j=1}^{\mu} \int_{U'_j} \varphi_j^2 (\tilde{f} - h'_j)^2 + \varphi_j^2 (D\tilde{f} - Dh'_j)^2 + |D\varphi_j|^2 (\tilde{f} - h'_j)^2 \\
&\le C \sum_{j=1}^{\mu} \left(\|\tilde{f} - h'_j\|^2_{W^{1,2}(D)} + \int_{U'_j} |D\varphi_j|^2 (\tilde{f} - h'_j)^2 \right) \\
&= O(\varepsilon)
\end{aligned}$$

for suitable $\delta_0 > 0$ with $k > k_0 > 0$, since $\varphi_j^2 \le 1$ and $|D\varphi_j|^2 \le \frac{C}{\delta^2}$. The last inequality explains in detail how (3.27) is obtained.

§4 Continuity theorems

Given a quasi-Lipschitz domain $D \subset\subset M^n$, we have defined the space $H(D)$ as the space of Sobolev functions in $W^{1,2}(D)$, which satisfy the boundary L^2 vanishing condition. In Corollary 3.2, we have proved the closure theorem, and established that $H(D)$ coincides with the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$.

When $W^{1,2}(M^n)$ is mentioned in the following, we assume that M^n itself has compact boundary without loss of generality, since the concerned domains D in M^n have compact closures, and M^n can be restricted to a smaller part while maintaining $D \subset\subset M^n$.

Proposition 4.1. *Let $g \in W^{1,2}(D)$, then $g \in H(D)$ if and only if g can be extended to $\tilde{g} \in W^{1,2}(M^n)$ with $\tilde{g} \equiv 0$ on $D_c := M^n \setminus \overline{D}$.*

Proof. (i) To prove the "if part", we choose $f_k \in C^\infty(M^n)$ with $\|\tilde{g} - f_k\|^2_{W^{1,2}(M^n)} \to 0$, as $k \to \infty$. Using the arguments in Step 1 of the proof of Theorem 3.1, we have $\int_{\partial D} f_k^2 \to 0$, i.e. $g \in H(D)$, where g is the restriction of $\tilde{g}$ into D.

(ii) On the other hand, suppose that $g \in H(D)$. We extend it to $\tilde{g} \in L^2(M^n)$ by letting $\tilde{g} \equiv 0$ on D_c. From Theorem 3.1, g is in the closure of $\mathcal{F}_{cpt}(D)$ in $W^{1,2}(D)$, i.e.$\forall \varepsilon > 0$ there exists $f \in \mathcal{F}_{cpt}(D)$ such that $\|f - g\|^2_{W^{1,2}(D)} < \varepsilon$. Extend f to $\tilde{f} \in C^\infty(M^n)$ by letting $\tilde{f} \equiv 0$ in D_c. Then $\|\tilde{f} - \tilde{g}\|^2_{W^{1,2}(M^n)} < \varepsilon$. Thus $\tilde{g} \in W^{1,2}(M^n)$. ∎

Based on this equivalence, we *identify* g with its extension $\tilde{g}$, for convenience. Namely, a Sobolev function of $H(D)$ is identified with the one of $W^{1,2}(M^n)$ with zero value on $M^n \setminus \overline{D}$.

Theorem 4.1 (Sobolev Continuity)**.** *Given a monotone C^0-deformation*

$$\mathcal{D} := \{D(t) \subset M^n \mid 0 \leq t \leq b\}$$

on M^n, defined by Definition 1.1. Denote the Sobolev space $H_t := H(D(t))$ (Definition 1.2). Then $\mathcal{D}$ satisfies the Sobolev continuity

$$\overline{\bigcup_{s<t} H_s} = H_t = \bigcap_{r>t} H_r. \tag{4.1}$$

Note that the sobolev continuity (4.1) is also valid for any of the three Sobolev spaces which are the closures of $\mathcal{F}_{cpt}(D(t))$, $\mathcal{F}(D(t))$ and $\mathcal{F}_0(D(t)) \cap W^{1,2}(D(t))$ in $W^{1,2}(D(t))$, since they are all identical with $H(D(t))$.

Proof. Step 1. Claim that $H_t = \bigcap_{r>t} H_r$ for $t \in [0, b]$. The inclusion "$\subset$" is obvious, since $H_t \subset H_r$ for $r > t$. We show the converse "$\supset$" as follows.

(i) Given $g \in \bigcap_{r>t} H_r$, we claim that $g \equiv 0$ on $D(t)_c := M^n \setminus \overline{D(t)}$, where $D(t)_c$ means the set of "exterior points" of $D(t)$ in M^n. Then by the equivalence of Proposition 4.1, we can identify $g \in H_r$ as $g \in W^{1,2}(M^n)$ with $g \equiv 0$ on $D(r)_c := M^n \setminus \overline{D(r)}$.

(ii) For V an open set in $D(t)_c$, with $\overline{V} \cap D(t) = \varnothing$, there exists $r' > t$ such that $\overline{V} \cap \overline{D(r')} = \varnothing$. Suppose not, there exists $r_n \searrow t$ and points $x_n \in \overline{V} \cap D(r_n)$. Consider a convergent subsequence, still denoted by $\{x_n\}$, which tends to $x_0 \in \overline{V}$. Given any $r > t$, there exists n_0 such that $r > r_n > t$ for $n > n_0$. Hence, $x_n \in D(r_n) \subset D(r)$ for $n > n_0$, and $x_0 = \lim_{n\to\infty} x_n \in \overline{D(r)}$. Thus $x_0 \in \bigcap_{r>t} \overline{D(r)} = \overline{D(t)}$. By (1.2), and $x_0 \in \overline{V} \cap \overline{D(t)}$, against the given assumption of $\overline{V} \cap \overline{D(t)} = \varnothing$.

(iii) Claim that $g \equiv 0$ on $D(t)_c : M^n - \overline{D(t)}$, which means $g = 0$ almost everywhere in $D(t)_c$. For an open set V with $\overline{V} \cap \overline{D(t)} = \varnothing$, there exists $r' > t$ such that $\overline{V} \cap D(r') = \varnothing$, by (i). Since $g \in H_{r'}$, we have that $g \equiv 0$ on V, for any open set $V \subset D(t)_c$. It means that $g \in W^{1,2}(M^n)$ with $g \equiv 0$ on $D(t)_c$. Since $D(t) \subset\subset M^n$, we apply the trace theorem (Theorem 3.1) to $D(t)$, obtaining $g \in W_0^{1,2}(D(t)) := H_t$. Thus the right equality $H_t = \bigcap_{r>t} H_r$ of (4.1) is proved.

Step 2. (i) Now we shall show the left equality: $\overline{\bigcup_{s<t} H_s} = H_t$ for $t \in [0, b)$. This is much easier. To show "$\subset$": Given $g \in \overline{\bigcup_{s<t} H_s}$. For any $\varepsilon > 0$, there exists $f \in \bigcup_{s<t} H_s$, such that $\|g - f\|_{H_b}^2 \leq \varepsilon/2$. Let $f \in H_s$ for some $s < t$.

(ii) By the definition of $H(D)$, $f \in H_s = H(D(s))$ means that there exist $f_k \in C^\infty(D(s)) \cup C^0(\overline{D(s)})$, such that $\int_{\partial D(s)} f_k^2 \to 0$ and $\|f_k - f\|_{W^{1,2}(D(s))}^2 \to 0$, as $k \to 0$. To claim that $g \in H_t$, we have to show that there exist $u_k \in C^\infty(D(t)) \cup C^0(\overline{D(t)})$, such that $\int_{\partial D(t)} u_k^2 \to 0$ and $\|u_k - g\|_{W^{1,2}(D(t))}^2 \to 0$, as $k \to 0$. But if we let $u_k = f_k$ on $D(s)$, and let $u_k = 0$ on $D(t) \setminus \overline{D}(s)$, u_k is not the smooth extension of f_k to $D(t)$. This is the difficulty of the problem. However, the trick is using Theorem 4.2, there exists $h \in \mathcal{F}_{cpt}(D(s))$ to approximate f in $W^{1,2}(D(s))$.

(iii) The smooth function h satisfies $\|f-h\|^2_{W^{1,2}(M^n)} < \varepsilon/2$. Furthermore, $h \in \mathcal{F}_{cpt}(D(s)) \subset \mathcal{F}_{cpt}(D(t))$. Therefore $\|g-h\|^2_{W^{1,2}(M^n)} < \varepsilon/2+\varepsilon/2 = \varepsilon$, and hence $g \in H_t$, as required.

Step 3. It remains to show $\overline{\bigcup_{s<t} H_s} \supset H_t$. Given $g \in H_t$. For any $\varepsilon > 0$, there exists $h \in \mathcal{F}_{cpt}(D(t))$ such that $\|g-h\|^2_{W^{1,2}(M^n)} < \varepsilon$. Let $K := \overline{\text{supp } h}$. For any $p \in K \subset D(t)$, there exists $s_p < t$ such that $p \in D(s_p) \subset D(t)$, since $D(t) = \bigcup_{s<t} D(s)$ by (1.2). Choose an open neighborhood N_p of p in $D(s_p)$. Evidently, $\bigcup\{N_p : p \in K\} \supset K$. As K is compact, there exist $p_1, \ldots, p_r$ such that $\bigcup\{N_{p_i} : i = 1, \ldots, r\} \supset K$. Hence $K \subset D(\overline{s})$ where $\overline{s} := \max\{s_{p_1}, \ldots, s_{p_r}\}$. Thus $h \in \mathcal{F}_{cpt}(D(\overline{s})) \subset H_{\overline{s}}$ with $\|g-h\|^2_{W^{1,2}(M^n)} < \varepsilon$. Therefore $g \in \overline{\bigcup_{s<t} H_s}$, and the proof of the Sobolev continuity (4.1) is completed. □

Theorem 4.2 (Eigenvalue Continuity)**.** *Given a monotone C^0-deformation*

$$\mathcal{D} := \{D(t) \subset M^n \mid 0 \le t \le b\}$$

on M^n, defined by Definition 1.1. Let $\lambda_k(t) := \lambda_k(D(t))$ be the eigenvalues of the given strongly elliptic operator L (see (1.5), (1.11) and (1.13)). Then each $\lambda_k(t)$ is continuous in t. Furthermore, $\lambda_k(t)$ is strictly decreasing in t.

Frid–Thayer [F-T] proved the eigenvalue continuity of the stability operator for monotone C^∞-deformations of domains $D(t)$ in M^n. In that case, the domains $D(t)$ are diffeomorphic to each other, not allowing topological change along t. Owing to their works, the arguments of their proof can be carried over to our case of C^0-deformations of quasi-Lipschitz domains with a given strong elliptic operator L defined on the domains. The reason is that the argument of their proof reduces essentially to the Sobolev continuity. For their smooth case, where the domains are diffeomorphic to each other, the Sobolev continuity is trivial. Nevertheless, it becomes nontrivial in our framework because of the topological change of the domains. The singularities occurring at glued points are also a non-negligible factor. We have to build up the celebrated closure theorem (Corollary 3.2) to support the Sobolev continuity, Theorem 4.1. These considerations explain why we must devote considerable effort in establishing Sobolev continuity before showing eigenvalue continuity.

In order to confirm the eigenvalue continuity for our case, we rewrite the proof of Frid-Thayer on the eigenvalue continuity here, in a formulation consistent with the framework of this paper.

Proof of Theorem 4.2. Step 1. (Right-continuity) We claim the right continuity of $\lambda_k(t)$. Given $r_i \searrow t$, i.e., $\lim_{i\to\infty} r_i = t$, and $t < r_i < r_{i-1}$, $\forall\, i = 1, 2, 3, \ldots$. Denote by $\{u_k(t)\}$ an orthonormal basis of eigenfunctions of $L_t := L(D(t))$ in H_t, corresponding to $\{\lambda_k(t)\}$. For each r_i, let

$$v_1(r_i), v_2(r_i), \ldots, v_k(r_i), \ldots \in H_{r_i} \subset H_b$$

be an orthonormal base of eigenfunctions of L_{r_i} on $D(r_i)$, which corresponds to $\lambda_j(r_i)$, $j = 1, 2, \ldots, k, \ldots$. Recall that $v_k(r_i) := 0$ on $D(b) \setminus D(r_i)$. For each j, $\lambda_j(r_i)$ is increasing in i (See (1.13)), and is bounded by $\lambda_j(t)$. Clearly, $\lambda_j(r_i) \to$ some $\alpha_j \in \mathbb{R}$, as $i \to \infty$. We will show that $\alpha_j = \lambda_j(t)$.

Step 2. Fix $j \in \mathbb{N}$. The set $\{v_j(r_1), v_j(r_2), \ldots\}$ is bounded in $L^2(D(b))$, since $\|v_j(r_i)\|_{L^2(D(b))} = 1$. Claim that it has a subsequence converging to v_j in H_t. We write $v(r_i) := v_j(r_i)$ at the moment by dropping the index "j" to simplify notations. By (1.7) and (1.9), we see that

$$\begin{aligned} B_L(v(r_i), v(r_i)) &= \int_{D(r_i)} \alpha^{k\ell} D_k v(r_i) D_\ell v(r_i) + c v(r_i)^2 \\ &\geq \overline{\alpha} \|Dv(r_i)\|^2_{L^2(D(b))} + \overline{c}, \end{aligned} \tag{4.2}$$

where $\overline{\alpha}$ and $\overline{c}$ are the integrals of $\alpha = \alpha(x)$ and $c = c(x)$ over $D(r_i)$. On the other hand,

$$B_L(v(r_i), v(r_i)) = \lambda_j(r_i) \|v(r_i)\|^2_{L^2(D(r_i))} = \lambda_j(r_i) \leq \lambda_j(t). \tag{4.3}$$

Combining (4.2)–(4.3), we see that $\|Dv(r_i)\|^2 < c$, independent of i, and hence there is a subsequence of $v(r_i)$, which converges to a function v_j in $L^2(D(b))$. Still denote the subsequence by $v_j(r_i)$, $i = 1, 2, 3, \ldots$.

Remark that $v(r_i)$ is regular on $D(r_i)$, i.e., $v(r_i) \in C^\infty(D(r_i)) \cap C^0(\overline{D(r_i)})$. Let $w_{i\ell} := v(r_i) - v(r_\ell)$, $i > \ell$, $D(r_i) \supset D(r_\ell)$. Computing $\|Dw_{i\ell}\|^2_{L^2(D(b))}$, and using the integration by parts, we see that

$$\begin{aligned}\int_{D(b)} |Dw_{i\ell}|^2 &= \int_{D(b)} D_h w_{i\ell} D_h w_{i\ell} \\ &= \int_{D(b)} D_h(w_{i\ell} D_h w_{i\ell}) - \int_{D(b)} w_{i\ell}(D_h^2 w_{i\ell}) \to 0 \end{aligned} \tag{4.4}$$

as $i, \ell \to \infty$, since the former is the boundary term and the latter tends to zero due to $w_{i\ell} \to 0$ in $L^2(D(b))$. Thus, $Dv(r_i)$ is a Cauchy sequence in $L^2(D(b))$. Let $D_h v(r_i) \to \xi_h$ in $L^2(D(b))$. Then for each $\varphi \in \mathcal{F}_{cpt}(D(b))$,

$$\langle\, \xi_h \;,\; \varphi \,\rangle = \lim_{i\to\infty} \langle\, D_h v(r_i) \;,\; \varphi \,\rangle = \lim_{i\to\infty} \langle\, v(r_i) \;, D_h \varphi \,\rangle = \langle\, v_j \;,\; D_h \varphi \,\rangle, \tag{4.5}$$

which shows that $\xi_h = D_h v_j$ and $v_j \in H_b$.

Step 3. We will show that v_j is the eigenfunction of $L_t := L(D(t))$ on $D(t)$ with eigenvalue α_j. Given $i \in \mathbb{N}$, $v(r_i)$, $v(r_{i+1})$, $\cdots \in H_{r_i} := H(D(r_i))$. The limit v_j is in H_{r_i} for any r_i. By the Sobolev continuity, $v_j \in \bigcap_{i=1}^{\infty} H_{r_i} = \bigcap_{r>t} H_r = H_t$, noting that $H_s \subset H_r$ for $s \le r$. Let H_t^2 be the closure of $\mathcal{F}(D(t)) := \{h \in C^\infty((D(t)) \cap C^0(\overline{D(t)})\,;\, h\big|_{\partial D(t)} = 0\}$ in $W^{2,2}(D(t))$. For any $w \in H_t^2$, we see that

$$\begin{aligned}\langle\, v_j \;,\; L_t w \,\rangle &= \lim_{i\to\infty} \langle\, v(r_i) \;,\; L_t w \,\rangle = \lim_{i\to\infty} B_L(v(r_i), w) \\ &= \lim_{i\to\infty} \lambda_j(r_i) \langle\, v(r_i) \;,\; w \,\rangle = \alpha_j \langle\, v_j \;,\; w \,\rangle,\end{aligned}$$

which means that $B_L(v_j, \varphi) = \alpha_j \langle v_j, \varphi\rangle$ for all $\varphi \in H_t$, since H_t^2 is dense in H_t. Therefore,

$$L_t v_j = \alpha_j v_j, \tag{4.6}$$

i.e., α_j is an eigenvalue of L_t on H_t.

Step 4. We now claim that α_j is the j-*th* eigenvalue of L_t on H_t. Clearly, $\lambda_j(t) \ge \lim_i \lambda_j(r_i) = \alpha_j$. Suppose $\lambda_1(t) > \alpha_1$. As α_1 is an eigenvalue of

L_t, α_1 must be some $\lambda_h(t)$ with $h > 1$. Thus we have

$$\lambda_1(t) \leq \lambda_h(t) = \alpha_1 \lneqq \lambda_1(t),$$

a contradiction. Hence $\lambda_1(t) = \alpha_1$. Inductively, we see $\lambda_j = \alpha_j$. Thus

$$\lim_{r_i \searrow t} \lambda_j(r_i) = \alpha_j = \lambda_j(t),$$

which shows the right-continuity.

Step 5. (Left-continuity) Let $\{u_k\}$ be an orthonormal basis of eigenfunctions defined in Step 1. Given $k \in \mathbb{N}$. Consider $j = 1, 2, \ldots, k$. By the first equality of Theorem 4.1 , i.e., $\overline{\bigcup_{s<t} H_s} = H_t$, there exist $s_1, s_2, \ldots \nearrow t$ and $\overline{v}_j(s_i) \in H_{s_i} \subset H_t$, $\forall\, i = 1, 2, \ldots$, such that $\overline{v}_j(s_i) \to u_j$ in $H_t \subset H_b$ as $i \to \infty$. Using min-max principle (1.12), we see that

$$\begin{aligned}
\lambda_k(s_i) &= \min_{V^k \subset H_{s_i}} \{ \max_{v \in V^k \cap S} B_L(v, v)\} && (\text{where} \dim V^k = k) \\
&\leq \max_{v \in \langle \overline{v}_1, \ldots, \overline{v}_k \rangle \cap S} B_L(v, v) && (\text{where } \overline{v}_j \equiv \overline{v}_j(s_i)) \\
&\xrightarrow{\text{as } i \to \infty} \max_{u \in \langle u_1, \ldots, u_k \rangle \cap S} B_L(u, u) = \lambda_k(t) \leq \lambda_k(s_i), \quad \forall\, i,
\end{aligned}$$

noting that S is the unit sphere in H_{s_i} or in H_t. Therefore

$$\lim_{s_i \nearrow t} \lambda_k(s_i) = \lambda_k(t).$$

The left-continuity is proved.

Step 6. (Strictness) It remains to show

$$s < t \quad \Rightarrow \quad \lambda_k(s) \gneqq \lambda_k(t), \ \forall\, k \in \mathbb{N}.$$

Suppose $\exists\, k > 0$ with $\lambda_k(s) = \lambda_k(t) \equiv \lambda$ for $s < t \leq b$. We shall construct $u \in H_s \subset H_t$ such that

$$L_s u = \lambda u \quad \text{and} \quad L_t u = \lambda u,$$

which is against $D(s) \subsetneqq D(t)$. Let $u_1, u_2, \ldots$ and $v_1, v_2, \ldots$ be orthonormal bases of eigenfunctions of L_s on H_s and of L_t on H_t, respectively. Choose $u = a_1 u_1 + \cdots + a_k u_k \in H_s \subset H_t$, with $\|u\| = 1$, such that

$\langle\, u, v_1 \,\rangle = \cdots = \langle\, u, v_{k-1} \,\rangle = 0$. Clearly, such a_i's are solvable, and $u \in C_0^\infty(D(s))$. Write

$$u = \langle\, u, v_k \,\rangle\, v_k + \langle\, u, v_{k+1} \,\rangle\, v_{k+1} + \cdots .$$

Then

$$\begin{aligned}\langle\, L_t u, u \,\rangle &= \lambda_k(t)\, \langle\, u, v_k \,\rangle^2 + \lambda_{k+1}(t)\, \langle\, u, v_{k+1} \,\rangle^2 + \cdots \\ &\geq \lambda_k(t)\, (\, \langle\, u, v_k \,\rangle^2 + \langle\, u, v_{k+1} \,\rangle^2 + \cdots) = \lambda_k(t) = \lambda.\end{aligned} \tag{4.7}$$

But

$$\begin{aligned}\langle\, L_s u, u \,\rangle &= \lambda_1(s) a_1^2 + \cdots + \lambda_k(s) a_k^2 \\ &\leq \lambda_k(s)\, (\, a_1^2 + \cdots + a_k^2 \,) = \lambda_k(s) = \lambda.\end{aligned} \tag{4.8}$$

Given $w \in H_t$, we know

$$\langle\, L_s u, w \,\rangle = \langle\, L_t u, w \,\rangle - E \neq \langle\, L_t u, w \,\rangle,$$

in general, although $I_s(u, w) = I_t(u, w)$. Here E is the boundary term $\int_{\partial D(s)} (D_\nu u) w$. By choosing $w = u \in H_s \subset H_t$, we have $w \equiv 0$ on $\partial D(s)$ and hence $E = 0$. Thus

$$\langle\, L_s u, u \,\rangle = \langle\, L_t u, u \,\rangle. \tag{4.9}$$

A combination of (4.7), (4.8) and (4.9) yields that $\lambda \geq \langle\, L_s u, u \,\rangle = \langle\, L_t u, u \,\rangle \geq \lambda$. It forces all the inequalities in (4.7) and (4.8) to be equalities. Hence $\lambda_1(s) = \lambda_2(s) = \cdots = \lambda_k(s) = \lambda$, and $L_s u = \lambda u$. On the other hand, $\lambda_l(t) \to \infty$ as $l \to \infty$. Looking at (4.7), there is an $m \geq 0$ such that $u = \langle\, u, v_k \,\rangle\, v_k + \cdots + \langle\, u, v_{k+m} \,\rangle\, v_{k+m}$, and each of $v_k, \ldots, v_{k+m}$ has the same eigenvalue λ. Thus

$$L_t u = \lambda u.$$

Now $u \in H_s \subset H_t$, $u \equiv 0$ on $D(t) - D(s)$. By the assumption that $D(t) - D(s) \neq \phi$, there is a nonempty open set U contained in $D(t) - D(s)$. Using Hopf's sphere theorem, we see that $u \equiv 0$ on $D(t)$, contradicting $\|u\| = 1$. $\square$

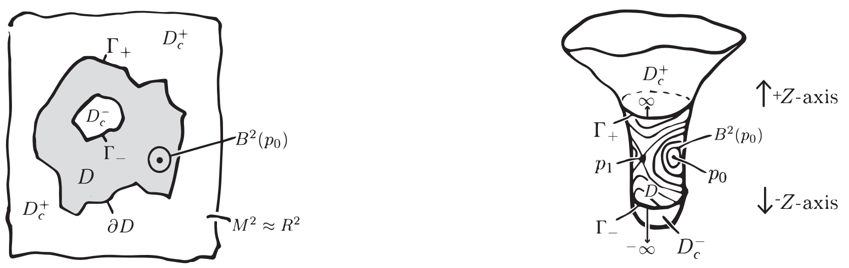
§5 The existence

We are ready to show the existence of C^0-deformations allowing topological changes as stated in Theorem A. Let the operator L on M^n and a quasi-Lipschitz domain D in M^n be given in Theorem A. For any $p_0 \in D$, we shall construct the required C^0- deformation on M^n, along which both Sobolev continuity and eigenvalue continuity of L hold, such that $D(0)$ is a small n-ball $B^n(p_0)$ and $D(b)$= the given D.

We first present two amusing examples in lower dimensions, before presenting the general construction of the C^0-deformation in the proof of Theorem A.

Example C. Consider an annular domain $D := S^1 \times I$ in the plane M^2 ($\approx \mathbb{R}^2$), where I:= the interval $[0, 1]$. Its boundary consists of Γ_+ and Γ_- which are the outer and the inner simple curves respectively. See Figure 5.1(a). Let $B^2(p_0)$ be a small 2-ball $\subset D \subset M^2$ centered at $p_0 \in D$ with radius δ.

We shall construct a C^0-deformation with $D(0) = B^2(p_0)$ and $D(b) = D$. To visualize the construction, we pull Γ_+ upward along the positive z-axis and pull Γ_- downward along the negative z-axis, as illustrated in Figure 5.1(b). The inner complementary domain D_c^- becomes a bowl-shaped region at the bottom of the cylinder, while the outer complementary domain D_c^+ lies above D. See Figure 5.1(b).



(a) the given Lipschitz domain D with one hole

(b) deforming $B^2(p_0)$ to one-hole domain D

Figure 5.1: Example C

Observe that D now appears as a region like a vertical cylinder, where D_c^+ extends to infinity. Moreover, $M^2 = D_c^- \cup \overline{D} \cup D_c^+$ is diffemorphic to $\mathbb{R}^2$. Let the small ball $D(0) = B^2(p_0)$ expand inside the annular domain D toward the opposite side of the cylinder. At the critical point p_1 of the function $h(p) := (dist_{M^2}(p, p_0))^2 - \delta^2$, the boundary $\partial D(t)$ develops a singularity where two boundary points become glued together at the critical point p_1 . (Compare this with Step 1 in the proof of Theorem A, which shall be provided later after Example D.)

The sector domain Λ then appears around p_1. See Figure 5.1(b). Indeed, $D(t)$ is the sublevel set of h. After the critical time $t > h(p_1)$, the domain $D(t)$ continues to expand until it reaches the two boundary circles $\Gamma_+ \cup \Gamma_- = \partial D$. Hence, at the end of the deformation, $D(b) = D$.

(a) the given Lipschitz domain D with two holes $D := \Omega \setminus (\Omega_1 \cup \Omega_2)$

(b) 1st step to visualize C^0-deformation on a two-hole domain D

Figure 5.2: Example D.

Example D. Consider a domain D in a plane M^2 with two holes, namely $D := \Omega\setminus\ \Omega_1 \cup \Omega_2$ where Ω is a large disk, and Ω_1, Ω_2 are two small disks compactly contained in Ω. See Figure 5.2(a). Let $\Gamma := \partial\Omega$, $\Gamma_1 := \partial\Omega_1$, and $\Gamma_2 := \partial\Omega_2$. As in Example C, we shall deform a given small 2-ball $B^2(p_0) = D(0)$ inside D into the entire domain $D = D(b)$. Note that the so-called 2-ball is exactly a disk. To visualize the deformation, pull the outer boundary circle Γ downward along the negative z-axis, while pulling Γ_1 and Γ_2 upward along the positive z-axis. See Figure 5.2(b). Then the domain D resembles a tree with a vertical trunk and two branches stretching upward into the sky. See Figure 5.2(c).

Starting from the small disk $B^2(p_0) = D(0)$, the sublevel sets $D(t)$ expand along the trunk as in Example C. The given small 2-ball $B^2(p_0) = D(0)$ on the trunk expands until a first singular transition occurs at the critical point p_1 of the distance function $h = h(p) := (dist_{M^2}(p, p_0))^2 - \delta^2$. At the critical level $t = h(p_1)$, two points on the ∂D become glued together at p_1. Compare Figure 5.2(c). A sector domain appears around p_1.

(c) deforming $B^2(p_0)$ to a two-hole domain D

(d) restore the deformation on the given planar D with two holes

Figure 5.2: Example D.

For $t > h(p_1)$, the domain $D(t)$ continues expanding with its upper boundary curve approaching toward the two upper branches. A second singular transition occurs at another critical point p_2 where $t = h(p_2)$. See Figure 5.2(c). Again, two boundary points of $D(t)$ become glued together, and another sector domain appears around p_2. After passing the second critical level, the expanding domains $D(t)$ eventually reach all three boundary components Γ, Γ_1 and Γ_2. Hence, $D(b) = D$ at the end of the deformation. It is interesting to see how $D(t)$ deforms in the originally given planar domain. We draw Figure 5.2(d) to restore the expanding process of $D(t)$.

We now present a general process of the C^0-deformation from a small n-ball to a given Lipschitz domain D.

Proof of Theorem A. For the first part of Theorem A, the proof has already been obtained from the continuity theorems, Theorems 4.1 and 4.2. It remains to prove the existence of the required C^0-deformation of domains.

Step 1. Given any $p_0 \in D$, and a small n-ball $B^n(p_0)$ centered at p_0 with radius $\delta > 0$. Consider $h\colon D \xrightarrow{C^\infty} \mathbb{R}^1$ defined by $h(p) := (\mathrm{dist}_{M^n}(p, p_0))^2 - \delta^2$ for all $p \in D$. We may assume by a slight modification of h that h is *nondegenerate* in the sense that the Hessian $\mathrm{Hess}(h)$ of h has no zero eigenvalue. This is possible since the set of nondegenerate functions is dense in $C^\infty(D, \mathbb{R})$, which is equipped with the compact-open topology. Let $\{p_0, p_1, p_2, \ldots, p_\mu\}$ be the set of critical points of h in D. Again, by a slight perturbation, we assume no p_j with

$h(p_j) = h(p_{j+1})$, $j = 0, 1, \ldots, \mu - 1$. Let

$$h(p_0) < h(p_1) < h(p_2) < \cdots < h(p_\mu).$$

Define

$$D(t) := \{p \in D \mid h(p) < t\}$$

the "sublevel set" of h at t. Then $D(0) = n$-ball $B^n(p_0)$. Write $t_j := h(p_j)$, $j = 1, 2, \ldots \mu$. Consider the sequence

$$0 < t_1 < c_1 < t_2 < \cdots < c_{\mu-1} < t_\mu < c_\mu < b.$$

We will show that

$$\mathcal{D} := \{D(t) \subset M^n \mid t \in [0, b]\}$$

is a monotone C^0-deformation on M^n where $D(0) = B^n(p_0)$, $D(b)$ is the given Lipschitz domain D in M^n, where $b := \sup\{h(p) \mid p \in D\}$.

Step 2. To ensure that all the critical points p_j of h are sitting in an interior domain D_0 compactly contained in D, we have to modify the metric of D around its boundary by stretching ∂D sufficiently far away, so that ∂D is disjoint from the closure of the sublevel set $D(c_\mu) := \{p \in D; h(p) < c_\mu\}$, where $t_\mu < c_\mu < b$. In fact, let N_ϵ be a tubular neighborhood of ∂D in D in the begining. Namely, let $N_\epsilon := \{p \in D; \zeta(p) := \mathrm{dist}_{M^n}(p, \partial D) < \epsilon\}$, $\epsilon > 0$ a small number. We remodel the metric g_{ij} of M^n in N_ϵ by

$$\widetilde{g}_{ij}(p) := (\Lambda \cdot (\zeta(p) - \epsilon)^2 + 1) g_{ij}(p)$$

with Λ sufficiently large. Then we can have $h(N_{\epsilon/2}) > c_\mu$, and $\{p_j; j = 1, 2, \ldots, \mu\} \subset D_0 := D \setminus \overline{N_{\epsilon/2}}$.

Step 3. Classical Morse theory tells us that D is homotopic to the cell-complex C_h, constituted by the family $\{\sigma_j \mid j = 0, 1, 2, \ldots, \mu\}$ of cells σ_j with $\gamma_j := \dim \sigma_j =$ the index of the Hessian Hess(h) of h at p_j, $j = 0, 1, 2, \ldots, \mu$. Furthermore, it says that any two sublevel sets $D(t)$ with $t_{j-1} < t < t_j$ are homotopic to each other along the integral curves of the gradient ∇h of h on D. At $t_j = h(p_j)$, consider $\widehat{D}_j := \sigma_j \bigcup_{\varphi_j} D(c_{j-1})$, where $\partial \sigma_j$ is diffeomorphic to the $(\gamma_j - 1)$-dimensional sphere $S^{\gamma_j - 1}$, and it is glued to $\partial D(c_{j-1})$ by a "diffeomorphic into" map $\varphi_j \colon \partial\sigma_j \to \partial D(c_{j-1})$. (See Figures 5.3 and Figures 5.4 for a special case.) As t increases from c_{j-1} to t_j, $\widehat{D}_j(t) := \sigma_j(t) \bigcup_{\varphi_j(t)} D(t)$ varies to become $D(t_j)$ with the γ_j-cell $\sigma_j(t)$ shrinked to the point p_j. To visualize the gluing, see also the following Step 4. Remark that $D(t_j)$ is a quasi-Lipschitz domain in M^n with glued point p_j, where the model space Ω is $D(c_{j-1})$ and the preglued set $\alpha^k = \partial\sigma_j \subset \partial D(c_{j-1}) = \partial\Omega$. Check Definition 2.4, if needed. The dimension $k = k(j)$ of α^k depends on j, and equals $\gamma_j - 1$, i.e., $k = \gamma_j - 1$.

Step 4. Clearly, the index γ_0 of $\mathrm{Hess}(h)$ is zero at p_o, since h is nondegenerate and attains minimum at p_0. Thus the sublevel set $D(0)$ is diffeomorphic to an n-ball $B^n(p_0)$. We illustrate the arguments developed in Step 3 by a simple case that $n = 3$, $j = 1$, $p_j = p_1$, and $\gamma_1 := \dim \sigma_1 = 1$. See Figure 5.4(a) and Example F. Remark that $D(0)$ is a 3-ball, which is illustrated in Figure 5.4(a) by the exterior 3-ball $B^3_c \cup p_\infty$ as before in Example A.

Here σ_1 is an open interval (q'_0, q'_1) and $\partial\sigma_1$ is the set of two points $\{q'_0, q'_1\}$. Let $\widehat{D}_1 := \sigma_1 \bigcup_{\varphi_1} D(0)$, which looks like a 3-ball $D(0)$ attached with a 1-hande $\sigma_1 = (q'_0, q'_1)$ with $\partial\sigma_1 = \{q'_0, q'_1\}$ glued on the boundary $\partial D(0)$ of the 3-ball $B^3_c \cup p_\infty$. See Figure 5.4(a). Here the model space Ω is the 3-ball $D(0)$, and the preglued set is $\alpha^0 := \partial\sigma_1 = \{q'_0, q'_1\}$. As t increases from 0 to $t_1 := h(p_1)$, the two points q'_0 and q'_1 get closer and closer to each other, and finally glued together at the critical point p_1 of h. See Figure 5.4(b), Figure 5.4(c) and Example F. By h nondegenerate at p_1, the shaded area $D(t_1)$ is locally a "sector domain" around p_1, and hence the sublevel set $D(t_1)$ is a quasi-Lipschitz domain in M^3. The topological structure of $D(t)$ changes at $t = t_1$. When t increases further from t_1 to c_1, $D(t)$ expands along ∇h to become $D(c_1)$, and the topology of $D(t)$ changes from a 3-ball to a solid torus $D(c_1)$.

Step 5. In summary, we have constructed a C^0-deformation $\mathcal{D}$ of enlarging quasi-Lipschitz domains in M^n. In fact, each domain $D(t)$ has smooth boundary, except at $t = t_j := h(p_j)$ for some $j = 1, 2, \ldots, \mu$, where a glued point p_j of $D(t_j)$ appears with preglued set $\alpha^k = \partial\sigma_j$, and $D(t_j)$ is a quasi-Lipschitz domain in M^n, satisfying the conditions of Definition 2.4. We note that $D(t_j)$ is a sector domain around p_j, because : (1) h is nondegenerate at p_j; (2) $D(t_j)$ tangents to the portion of the tangent space $T_{p_j}(M^n)$ at p_j on which Hess(h) is negative. This is due to the well-known Morse lemma (see John Milnor: Morse theory).

Step 6. As $t > h(p_\mu)$, the sublevel set $D(t)$ of h continuously expands, and it reaches the boundary ∂D at $t = b$ with the modification of metric $g_{i,j}$ in Step 2, since $b = sup\{h(p); p \in D\}$. Hence we obtain the required C^0-deformation $\mathcal{D} := \{D(t); t \in [0, b]\}$ with $D(0) = B^n(p_0)$, and $D(b)$ is the given quasi-Lipschitz domain D in M^n. This completes the proof of Theorem A. □

In Examples A and B of Section 2, we illustrate two typical types of quasi-Lipschitz domains: set-glued and point-glued. Now we realize the process of topological changes of the quasi-Lipschitz domains in the C^0-deformations around the critical points of the distance function h, based on these two types of quasi-Lipschitz domains.

Example E. Consider $n = 3$, $p_j = p_1$, at which the index is 2, i.e., the corresponding cell σ_1 has dimension $\gamma_1 = 2$, and $\sigma_1 \approx$ 2-disk. Here the sample space Ω is the sublevel set $D(0) \approx B^3(p_0)$ with the pre-glued set $\alpha^k = \alpha^1 = \partial\sigma_1 =$ the circle $S^1 \subset \partial D(0) \approx \partial B^3(p_0) \approx$ a 2-sphere S^2. As in Example A, we exchange the interior and exterior of $B^3(p_0)$, by the standard inversion, letting $p_0 = p_\infty$ at infinity (see Figure 5.3(a)). In Figure 5.3(a), $\widehat{D}_1 =$ the 2-disk σ_1 glued to $B^3(p_\infty)$ on its boundary $\partial B^3(p_\infty)$. As t increases from 0 to t_1, the 2-disk $\sigma_1(t)$ shrinks to the point p_1 (see Figures 5.3(b) and 5.3(c)). Finally, we obtain $D(t_1)$ as the whole 3-space $\mathbb{R}^3$, compactified by adding the point p_∞ at infinity, from which two small balls B_1 and B_2 are dug out, while $\overline{B}_1 \cap \overline{B}_2 = \{p_1\}$.

Example F. Another example to visualize the construction of the C^0-deformation in the proof of Theorem A is relying on Example B. Consider $n = 3$, $p_j = p_1$, at which the index γ_1 of Hess(h) is 1, $\sigma_1 =$ 1-cell $=$ the interval (q_0', q_1') with $\partial\sigma_1 =$ two points $\{q_0', q_1'\}$ glued on $\partial D(0) \approx S^2$ (see Figure 5.4(a)). As t increases from 0 to t_1, the interval $\sigma_1(t) = (q_0'(t), q_1'(t))$ shrinks into the critical point p_1 (see Figures 5.4(b) and 5.4(c)), which is the glued point of the quasi-Lipschitz

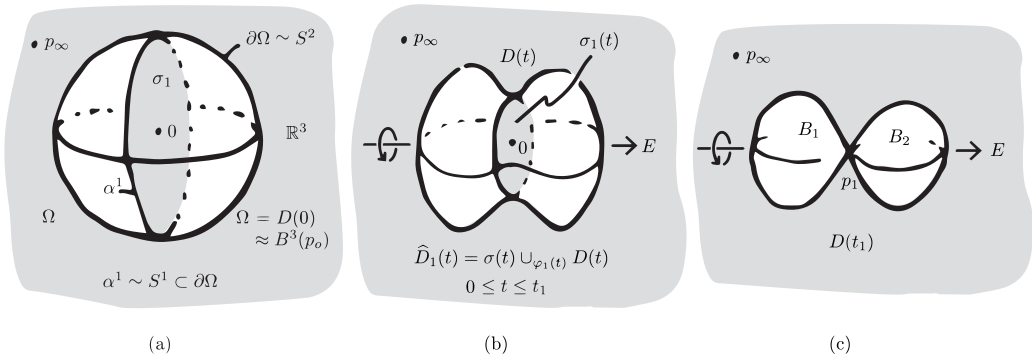


Figure 5.3: Example E - index of $h = 2$; construction of C^0-deformation, changing topology from (a) to (c).

domain $D(t_1)$. Note that $D(t_1)$ is the whole 3-space $\mathbb{R}^3$, compactified with the point p_∞ at infinity, from which a solid torus $V^2 \times S^1$ is dug out, i.e., $D(t_1) \approx B^3 \setminus (V^2 \times S^1)$, where V^2 is a planar sector. See Figure 5.4(c).

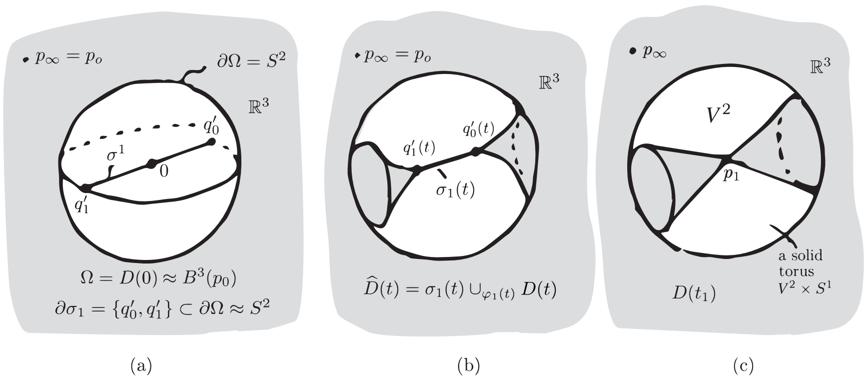


Figure 5.4: Example F - index of $h = 1$; construction of C^0-deformation, changing topology from (a) to (c).

Proof of Theorem B. We prove the global Morse index theorem in the following.

Step 1. Given $\varphi \in H_t = H(D(t))$, we expand

$$\varphi = a_1\, u_1 + a_2\, u_2 + \cdots \tag{5.1}$$

in $L^2(D(t))$, where the eigenfunctions $u_k \in \mathcal{F}_0(D(t))$ are the orthonormal basis. Then for any $g \in H_t$,

$$B_L(\varphi, g) = \langle\, a_1\, \lambda_1 u_1 + a_2\, \lambda_2 u_2 + \cdots\, ,\; g\, \rangle = 0. \tag{5.2}$$

if and only if $a_i = 0$, $\forall\, i$ with $\lambda_i(t) \neq 0$. The "if" part is evident, and the "only if" part can be seen by letting $g = u_i$. Thus there exists a (non trivial) Jacobi field on $D(t)$ if and only if B_L on $D(t)$ has a zero eigen value. Hence, the nulity $\nu(t)$ is the multiplicity of the zero eigenvalues of B_L.

Step 2. Let the eigenvalues λ_j on $D(t)$ be written in the form

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k < 0 \leq \lambda_{k+1} \leq \lambda_{k+2} \leq \cdots \to \infty. \tag{5.3}$$

When $D(s)$ enlarges from 0 to t, each of $\lambda_1, \lambda_2, \ldots, \lambda_k$ decreases and continuously passes through zero to become negative. We see these by (1.13) and Theorem 4.2. It is evident that $\sum_{0 \leq s < t} \nu(t) = k$, since as s varies from 0 to t, there occurs a Jacobi field on $D(s)$, if and only if some $\lambda_j(s)$ equals 0, for j with $1 < j \leq k$, counting the multiplicity.

Step 3. On the other hand, the Morse index $Ind(D(t))$ is definded by the maiximal dimension of linear subspaces of $H(D(t))$ on which B_L is negatively definite. It means that $Ind(D(t))$ = number of negative eigeinvalues of B_L. By the sequence (5.3), in which the eigenvalues λ_j are linearly ordered, it is clear that $Ind(D(t)) = k = \sum_{0 \leq s < t} \nu(t)$. This completes the proof. □

In an earlier paper with C.C. Lin [H-L], we introduced the notion of *extremal domains* to initiate the study of Jacobi fields on CMC hypersurfaces M^n immersed in $\mathbb{R}^{n+1}$. In [Hw] we obtained a global Morse index theorem of generalized Lipschitz domains to answer the distribution question of Jacobi fields. The following Theorem C improves the results of [Hw] by constructing a C^0-deformation of quasi-Lipschitz domains on the CMC hypersurface M^n, for which the global Morse index theorem is also valid.

As in Theorems A and B, the C^0-deformation allows topological changes of the domains while preserving Sobolev continuity and the continuity of eigenvalues along t.

Theorem C. *Let M^n be a hypersurface of constant mean curvature (CMC) immersed in $\mathbb{R}^{n+1}$, $D \subset M^n$ be a quasi-Lipschitz domain in M^n, and L be the stability operator*

$$Lf := -\Delta_M f - \|B\|^2 f, \quad \forall\, f \in \mathcal{F}(D) \tag{5.4}$$

extended to the Sobolev sense (see [Hw] for details), where $\Delta_M := (*)_{ii}$ *means the Laplacian operator relative to the metric of* M^n*, and* $\|B\|^2$ *denotes the length of the second fundamental form of* M^n *in* $\mathbb{R}^{n+1}$*. Given* $p_0 \in D$*. Then there exists a* C^0*-deformation* $\mathcal{D}$ *of enlarging quasi-Lipschitz domains in* D*, where*

$$\mathcal{D} := \{D(t) \subset D \mid 0 \leq t \leq b\},$$

$D(0) = B^n(p_0)$*, and* $D(b) = D$*. The index* $\mathrm{Ind}(D)$ *of* L *on* D *can be counted by* (1.15)*, where* $\nu(D(t))$ *is the nullity of* L *on* $D(t)$*. Indeed, given any* C^0*-deformation* $\mathcal{D}$ *of enlarging quasi-Lipschitz domains in* M^n*, the Sobolev continuity in* t *holds(see* (1.14)*), the eigenvalues* $\lambda_k(t)$ *of* L *are continuous in* t*, and the Morse index theorem* (1.15) *is valid. Furthermore, instead of the Dirichlet case considered in* (5.4)*, we consider the stability operator*

$$\widetilde{L}g := -\Delta_M g - \|B\|^2 g - \frac{1}{|D|}\int_D Lg, \quad \forall\, g \in \mathcal{G}(D)$$

for the case with volume constraint, where

$$\mathcal{G}(D) := \left\{ g \in \mathcal{F}(D) \;\Big|\; \int_D f\, dM = 0 \right\}.$$

The previous assertions in the Dirichlet case are also valid for the case with volume constraint.

Proof of Theorem C. Since the construction of the C^0-deformation in the proof of Theorem A involves only topology and the differentiable structure, it is still applicable to CMC hypersurfaces in $\mathbb{R}^{n+1}$. To show the global Morse index theorem (1.15), we need Sobolev continuity and eigenvalue continuity proved in Theorems 4.1 and 4.2, as well as the arguments in Sections 4.1 and 4.2 in [Hw]. These are now established and available. The proof of Theorem C is thus completed. □

§6 Concluding remarks

(i) In 1965, Smale [Ss] extended the Morse index theorem on geodesics to smooth domains enlarging in manifolds. The deformation $\mathcal{D}$ was assumed to be C^∞, meaning that all $D(t) \subset M^n$ must be of the same topological type. Hence the initial domain $D(0)$ of Smale's "ϵ-type" must share the topology same as the final domain $D = D(b)$. For example, if M^2 is the cylinder $S^1 \times \mathbb{R}^1$, and D a ring domain in M^2, the ϵ-type initial domain $D(0)$ must be also a ring domain $S^1 \times (-\varepsilon', \varepsilon')$, which is very thin with $\varepsilon' = O(\varepsilon)$. Theorem B in this paper removes the C^∞-deformation restriction and extends the Morse index theorem to a broader class of C^0-deformations where the topology of $D(t)$ is allowed to change along t. Thus, the initial domain $D(0)$ can be arbitrary, such as a ball $B^n(p_o)$ around a point p_o, and the final domain $D(b)$ can be a given quasi-Lipschitz domain of any shape. Furthermore, given two arbitrary quasi-Lipschitz domains D and D' on a manifold M^n, we can deform D to a small n-ball $B^n(p_0)$ by shrinking and then deform the n-ball to D' by expanding. Therefore, we obtain the existence of a C^0-deformation on which the Morse index theorem holds, with $D(0) = D$, $D(b) = D'$.

(ii) The proof of the Sobolev continuity in the smooth setting is trivial. Following Smale's smooth framework, Frid-Thayer [F-T] proved the eigenvalue continuity based on the Sobolev continuity, and a Morse index theorem in an abstract form. See Remark 3.2 in [Hw].

(iii) The work [Hw] of our previous paper was focused on generalized Lipschitz domains deforming in CMC hypersurfaces of $\mathbb{R}^{n+1}$. The streamlined proof of the Sobolev continuity, i.e. Theorem 4.1, in the present paper improves the complicated arguments illustrated in the proof of Theorem S of [Hw], and facilitates better understanding. They also help to clarify the tedious reasoning in certain contexts of [Hw].

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